

APPENDIX TO PAPER: A GEOMETRIC LEVEL-SET FORMULATION OF A PLASMA-SHEATH INTERFACE

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ABSTRACT. In this paper, we present appendices employed in the paper "A geometric level-set formulation of a plasma-sheath interface" by the authors.

APPENDIX A. Local existence of sheath solutions

In this appendix, we present a series of a priori estimates for the approximate solutions constructed in *Step 0 - Step 7* in Section 7 and then give the proof Theorem 7.3.

A.1. Basic a priori estimates. In this part, we give *a priori* estimates for the approximate solutions constructed in *Step 0 - Step 7*.

A.1.1. A priori estimates for Step 1. In Lemmas A1-A3, we will give a proof of the existence, uniqueness and regularity for n as given in *Step 1* of Section 7.2.

We first consider the equation for a characteristic curve. For given (\mathbf{x}, t) ,

$$(A.1) \quad \partial_s \chi(s, t, \mathbf{x}) = \mathbf{v}(\chi(s, t, \mathbf{x}), s), \quad \chi(t, t, \mathbf{x}) = \mathbf{x}, \quad 0 \leq s \leq T.$$

In what follows, we will use calculus type estimates for the Hölder seminorm. For $f_i \in C^{0,\gamma}(\bar{\Lambda}_s(T; \mathbf{v}))$ $i = 1, 2$, we have

$$(A.2) \quad [[f_1 f_2]]_{0,\gamma} \leq [[f_1]]_{0,\gamma} |||f_2|||_0 + |||f_1|||_0 [[f_2]]_{0,\gamma},$$

$$(A.3) \quad [[e^f]]_{0,\gamma} \leq e^{|||f|||_0} [[f]]_{0,\gamma}.$$

Here $[[\cdot]]_{0,\gamma}$ and $|||\cdot|||_0$ denote the Hölder and *esssup* norms defined on the same space-time region.

In the following Lemma, we use simplified notation for balls in \mathbb{R}^2 :

$$B_1 := B(0, r_b + 3K_0 \delta^* T_0) \quad \text{and} \quad B_2 := B(0, r_b + 6K_0 \delta^* T_0).$$

Lemma A.1. *There exists a sufficiently small constant $T_0 > 0$ and a unique solution χ to the equation (A.1) satisfying the following estimates: For $0 < T \leq T_0$, $\mathbf{v} \in \mathcal{B}(T)$,*

- (1) *The forward characteristic curve $\chi(s, 0, \mathbf{x})$, $s \geq 0$, $\mathbf{x} \in \Omega_s^1(0; \mathbf{v}) \subset (B_1 - \Omega_0)$ hits the target boundary $\partial\Omega_0$ and the ion-density in the region $\Lambda_s^1(T; \mathbf{v})$ is given by*

$$n(\chi(t, 0, \mathbf{x}), t) = n_0(\mathbf{x}) \exp \left(- \int_0^t (\nabla \cdot \mathbf{v})(\chi(s, 0, \mathbf{x}), s) ds \right), \quad \mathbf{x} \in \Omega_s^1(0; \mathbf{v}).$$

- (2) $\chi(s, t, \mathbf{x}) \in C^{1,\gamma}([0, T] \times [0, T] \times \mathbb{R}^2)$ and $\sup_{s,t,\mathbf{x}} \max_{i,j=1,2} |\partial_{x_j} \chi^i| \leq 2$.

- (3) Suppose that $\mathbf{v}_i \rightarrow \mathbf{v}$ in $C^{1,\gamma}(\bar{\Lambda}(T))$ and let χ_i and χ be the characteristic curves corresponding to \mathbf{v}_i and \mathbf{v} respectively. Then for $(\mathbf{x}, t) \in \Lambda_s^1(T; \mathbf{v})$,

$$\chi_i(\cdot, t, \mathbf{x}) \rightarrow \chi(\cdot, t, \mathbf{x}) \quad \text{in } C^{1,\gamma}([0, T]).$$

- (4) $\alpha(\mathbf{x}, t) := \chi(0, t, \mathbf{x})$ is Lipschitz continuous in $(\mathbf{x}, t) \in \Lambda(T)$ with a Lipschitz constant 4, i.e.

$$|\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, s)| \leq 4|(\mathbf{x}, t) - (\mathbf{y}, s)|.$$

Proof. (i) It follows from the dissipative condition ($\mathcal{D}2$) in the definition of $\mathcal{B}(T)$, we have

$$\mathbf{v}(\mathbf{x}, t) \cdot \mathbf{x} \leq -\frac{\eta_0}{2}|\mathbf{x}|^2, \quad (\mathbf{x}, t) \in (B_2 - \Omega_0) \times [0, T].$$

Then we have for $(\mathbf{x}, t) \in (B_2 - \Omega_0) \times [0, T]$,

$$\frac{d}{ds}|\chi(s, t, \mathbf{x})|^2 = 2\langle \mathbf{v}(\chi(s, t, \mathbf{x}), s), \chi(s, t, \mathbf{x}) \rangle \leq -\eta_0|\chi(s, t, \mathbf{x})|^2.$$

Here $\langle \cdot, \cdot \rangle$ denotes the standard inner product in \mathbb{R}^2 . Hence the characteristic $\chi(s, t, \mathbf{x})$ satisfies

$$|\chi(s, 0, \mathbf{x})| \leq e^{-\frac{\eta_0 s}{2}}|\chi(0, 0, \mathbf{x})| = e^{-\frac{\eta_0 s}{2}}|\mathbf{x}|.$$

So $\chi(s, 0, \mathbf{x})$ has decreasing magnitude and must hit the target at some positive s .

Let $T \leq T_0$ and we define the subregions $\Lambda_s^1(T; \mathbf{v})$, $\Omega_s^1(T; \mathbf{v})$ of $\Lambda(T)$ and $\Omega_s(0)$ as in *Step 1* of Section 7.2.1. Then the characteristic curve $\chi(s, 0, \mathbf{x})$, $(\mathbf{x}, 0) \in \Omega_s^1(0) \times \{t = 0\}$ hits the target boundary $\partial\Omega_0$ and will provide the ion density n at the target boundary, i.e.,

$$n(\chi(t, 0, \mathbf{x}), t) = n_0(\mathbf{x}) \exp\left(-\int_0^t (\nabla \cdot \mathbf{v})(\chi(s, 0, \mathbf{x}), s) ds\right), \quad \mathbf{x} \in \Omega_s^1(0; \mathbf{v}).$$

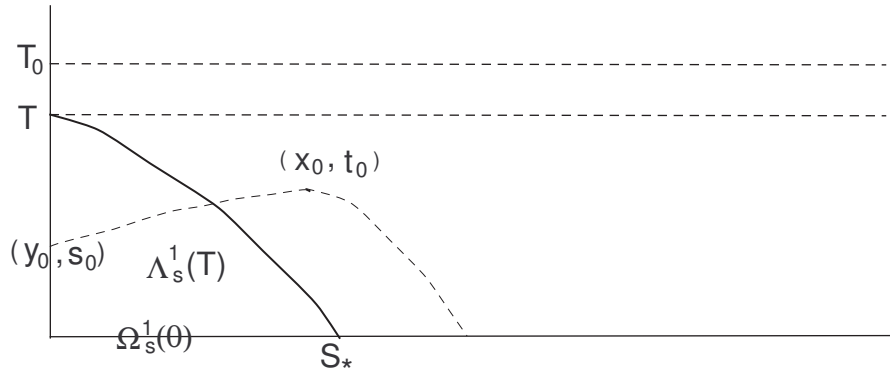


FIGURE 1. Schematic diagram of the geometry of characteristic curves

Remark A.1. We briefly summarize the geometry of characteristic curves in the space-time region $\Lambda(T)$ (see Figure 5).

The region $\Lambda_s^1(T)$ will be completely covered by the characteristic curves $\chi(s, 0, \mathbf{x})$, $(\mathbf{x}, 0) \in \Omega_s^1(0; \mathbf{v}) \times \{t = 0\}$ and they are pointing toward the target for positive s . On the other hand,

all backward characteristic curves $\chi(s, t, \mathbf{x}), 0 \leq s \leq t, (\mathbf{x}, t) \in \Lambda(T) - \Lambda_s^1(T)$ will either hit the initial region $(B(0, 3\delta^*) - \Omega_0) \times \{t = 0\}$ at $s = 0$ or the target boundary $\partial\Omega_0$ at some $s \in [0, t]$ (see Figure 5). However the latter situation will not happen, for example, suppose the backward characteristic curve $\chi(s, t_0, \mathbf{x}_0), 0 \leq s \leq t_0, (\mathbf{x}_0, t_0) \in \Lambda(T) - \Lambda_s^1(T)$ hits the target boundary at $s = s_0$ at \mathbf{y}_0 :

$$\mathbf{y}_0 := \chi(s_0, t_0, \mathbf{x}_0).$$

Then forward characteristic curve $\chi(s, s_0, \mathbf{y}_0), s \in [s_0, t_0]$ will have the same image of a trajectory as $\chi(s, t_0, \mathbf{x}_0), s \in [s_0, t_0], (\mathbf{x}_0, t_0) \in \Lambda(T) - \Lambda_s^1(T)$, but this is impossible since by the strong dissipation assumption D2 in the Definition 7.1, no forward characteristic curves will be issued from the target boundary.

(ii) The first part of the proof for (1) follows from the standard theory of ordinary differential equations. In fact we gain regularity in the s -variable, i.e.,

$$\chi(\cdot, t, \mathbf{x}) \in C^{2,\gamma}([0, T]).$$

We differentiate (A.1) with respect to x_j to get

$$\begin{cases} \partial_s \partial_{x_j} \chi^k(s, t, \mathbf{x}) = \nabla v_k(\chi(s, t, \mathbf{x}), s) \cdot \partial_{x_j} \chi(s, t, \mathbf{x}), & 0 \leq s \leq T \quad k, j \in \{1, 2\}, \\ \partial_{x_j} \chi^k(t, t, \mathbf{x}) = \delta_{jk}. \end{cases}$$

Here δ_{jk} is a Kronecker delta function and χ^k is the k -th component of χ , $k=1,2$.

We integrate the above equation along the characteristic curve χ to see

$$\partial_{x_j} \chi^k(\xi, t, \mathbf{x}) = \delta_{jk} - \int_{\xi}^t \nabla v_k(\chi(s, t, \mathbf{x}), s) \cdot \partial_{x_j} \chi(s, t, \mathbf{x}) ds.$$

The above relation implies

$$\sup_{s,t,\mathbf{x}} \max_{k,j=1,2} |\partial_{x_j} \chi^k| \leq 1 + 6K_0\delta^*(t - \xi) \sup_{s,t,\mathbf{x}} \max_{k,j=1,2} |\partial_{x_j} \chi^k|.$$

Since $t - \xi \leq T \ll 1$, we have

$$\sup_{s,t,\mathbf{x}} \max_{k,j=1,2} |\partial_{x_j} \chi^k| \leq 2.$$

(iii) Consider the equations for χ_i and χ : For $(\mathbf{x}, t) \in \Lambda_s^1(T; \mathbf{v})$,

$$\begin{cases} \partial_{\xi} \chi_i(\xi, t, \mathbf{x}) = \mathbf{v}_i(\chi_i(\xi, t, \mathbf{x}), \xi), & \text{and} \\ \chi_i(t, t, \mathbf{x}) = \mathbf{x}, \end{cases} \quad \begin{cases} \partial_{\xi} \chi(\xi, t, \mathbf{x}) = \mathbf{v}(\chi(\xi, t, \mathbf{x}), \xi), \\ \chi(t, t, \mathbf{x}) = \mathbf{x}. \end{cases}$$

We use the above equations to calculate $\chi_i(\xi, t, \mathbf{x}) - \chi(\xi, t, \mathbf{x})$, and integrate in ξ from $\xi = s$ to $\xi = t$ to get

$$\begin{aligned} \chi_i(s, t, \mathbf{x}) - \chi(s, t, \mathbf{x}) &= - \int_s^t \left(\mathbf{v}_i(\chi_i(\xi, t, \mathbf{x}), \xi) - \mathbf{v}(\chi(\xi, t, \mathbf{x}), \xi) \right) d\xi \\ &= - \int_s^t \left(\mathbf{v}_i(\chi_i(\xi, t, \mathbf{x}), \xi) - \mathbf{v}_i(\chi(\xi, t, \mathbf{x}), \xi) \right) d\xi \\ &\quad - \int_s^t \left(\mathbf{v}_i(\chi(\xi, t, \mathbf{x}), \xi) - \mathbf{v}(\chi(\xi, t, \mathbf{x}), \xi) \right) d\xi. \end{aligned} \tag{A.4}$$

Here we used $\chi_i(t, t, \mathbf{x}) = \chi(t, t, \mathbf{x}) = \mathbf{x}$ and note that

$$\begin{aligned}
 & \int_s^t \left(\mathbf{v}_i(\chi_i(\xi, t, \mathbf{x}), \xi) - \mathbf{v}_i(\chi(\xi, t, \mathbf{x}), \xi) \right) d\xi \\
 &= \int_s^t \int_0^1 \partial_{s_1} \mathbf{v}_i \left(\chi(\xi, t, \mathbf{x}) + s_1(\chi_i(\xi, t, \mathbf{x}) - \chi(\xi, t, \mathbf{x})), \xi \right) ds_1 d\xi \\
 &= \int_s^t \int_0^1 \nabla_{\mathbf{x}} \mathbf{v}_i \left(\chi(\xi, t, \mathbf{x}) + s_1(\chi_i(\xi, t, \mathbf{x}) - \chi(\xi, t, \mathbf{x})), \xi \right) \cdot \left(\chi_i(\xi, t, \mathbf{x}) - \chi(\xi, t, \mathbf{x}) \right) ds_1 d\xi.
 \end{aligned}
 \tag{A.5}$$

We now take the \mathbb{R}^2 -norm in (A.4) and use (A.5) to see

$$\begin{aligned}
 & |\chi_i(s, t, \mathbf{x}) - \chi(s, t, \mathbf{x})| \\
 & \leq |||\nabla \mathbf{v}_i|||_{0, \bar{\Lambda}(T)} \int_s^t |\chi_i(\xi, t, \mathbf{x}) - \chi(\xi, t, \mathbf{x})| d\xi + |||\mathbf{v}_i - \mathbf{v}|||_{0, \bar{\Lambda}(T)} (t - s).
 \end{aligned}$$

Note that Gronwall's inequality yields

$$\begin{aligned}
 & |\chi_i(s, t, \mathbf{x}) - \chi(s, t, \mathbf{x})| \\
 & \leq |||\mathbf{v}_i - \mathbf{v}|||_{0, \bar{\Lambda}(T)} (t - s) \left(1 + |||\nabla \mathbf{v}_i|||_{0, \bar{\Lambda}(T)} (t - s) e^{|||\nabla \mathbf{v}_i|||_{0, \bar{\Lambda}(T)} (t - s)} \right).
 \end{aligned}
 \tag{A.6}$$

By hypothesis (2) of this lemma, we have $\mathbf{v}_i \rightarrow \mathbf{v}$ in $C^{1, \gamma}(\bar{\Lambda}(T))$ as $t \rightarrow \infty$, and this implies from (A.6) that

$$\|\chi_i(s, t, \mathbf{x}) - \chi(s, t, \mathbf{x})\|_{0, [0, T]} \rightarrow 0 \quad \text{as } i \rightarrow \infty.
 \tag{A.7}$$

Next we show

$$\|\partial_s \chi_i(\cdot, t, \mathbf{x}) - \partial_s \chi(\cdot, t, \mathbf{x})\|_{0, [0, T]} \rightarrow 0 \quad \text{as } i \rightarrow \infty.
 \tag{A.8}$$

Note that (A.1) implies

$$\begin{aligned}
 & \|\partial_s \chi_i(\cdot, t, \mathbf{x}) - \partial_s \chi(\cdot, t, \mathbf{x})\|_{0, [0, T]} \\
 & \leq \|\mathbf{v}_i(\chi_i(\cdot, t, \mathbf{x}), \cdot) - \mathbf{v}_i(\chi(\cdot, t, \mathbf{x}), \cdot)\|_{0, [0, T]} + \|\mathbf{v}_i(\chi(\cdot, t, \mathbf{x}), \cdot) - \mathbf{v}(\chi(\cdot, t, \mathbf{x}), \cdot)\|_{0, [0, T]} \\
 & \leq |||\nabla \mathbf{v}_i|||_{0, \bar{\Lambda}(T)} \|\chi_i(\cdot, t, \mathbf{x}) - \chi(\cdot, t, \mathbf{x})\|_{0, [0, T]} + |||\mathbf{v}_i - \mathbf{v}|||_{0, \bar{\Lambda}(T)} \rightarrow 0 \quad \text{as } i \rightarrow \infty.
 \end{aligned}$$

We use (A.8) to show

$$[\chi_i(\cdot, t, \mathbf{x}) - \chi(\cdot, t, \mathbf{x})]_{0, \gamma, [0, T]} \rightarrow 0, \quad \text{as } i \rightarrow \infty.
 \tag{A.9}$$

By direct calculation we have

$$\begin{aligned}
 & \frac{|(\chi_i - \chi)(s_1, t, \mathbf{x}) - (\chi_i - \chi)(s_2, t, \mathbf{x})|}{|s_1 - s_2|^\gamma} \\
 & \leq \|\partial_s \chi_i(\cdot, t, \mathbf{x}) - \partial_s \chi(\cdot, t, \mathbf{x})\|_{0, [0, T]} |s_1 - s_2|^{1-\gamma} \\
 & \leq \|\partial_s \chi_i(\cdot, t, \mathbf{x}) - \partial_s \chi(\cdot, t, \mathbf{x})\|_{0, [0, T]} T^{1-\gamma} \rightarrow 0 \quad \text{as } i \rightarrow \infty.
 \end{aligned}$$

Next we show

$$[\partial_s \chi_i(\cdot, t, \mathbf{x}) - \partial_s \chi(\cdot, t, \mathbf{x})]_{0, \gamma, [0, T]} \rightarrow 0, \quad \text{as } i \rightarrow \infty.
 \tag{A.10}$$

It follows from (A.1) and (A.2) that

$$\begin{aligned}
 & [\partial_s \chi_i(\cdot, t, \mathbf{x}) - \partial_s \chi(\cdot, t, \mathbf{x})]_{0, \gamma, [0, T]} \\
 & \leq [\mathbf{v}_i(\chi_i(\cdot, t, \mathbf{x}), \cdot) - \mathbf{v}_i(\chi(\cdot, t, \mathbf{x}), \cdot)]_{0, \gamma, [0, T]} + [\mathbf{v}_i(\chi(\cdot, t, \mathbf{x}), \cdot) - \mathbf{v}(\chi(\cdot, t, \mathbf{x}), \cdot)]_{0, \gamma, [0, T]}
 \end{aligned}$$

$$\leq |||\nabla \mathbf{v}_i|||_{0,\gamma,\bar{\Lambda}(T)} ||\chi_i(\cdot, t, \mathbf{x}) - \chi(\cdot, t, \mathbf{x})||_{0,\gamma,[0,T]} + |||\mathbf{v}_i - \mathbf{v}|||_{0,\gamma,\bar{\Lambda}(T)} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

Here we used (A.7) and (A.9).

Finally we combine the estimates (A.7) - (A.10) to get

$$|||\chi_i(s, t, \mathbf{x}) - \chi(s, t, \mathbf{x})|||_{1,\gamma,[0,T]} \rightarrow 0 \quad \text{as } i \rightarrow \infty.$$

(iv) By the triangle inequality, we have

$$\frac{|\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, s)|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|} \leq \frac{|\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, t)|}{|\mathbf{x} - \mathbf{y}|} + \frac{|\alpha(\mathbf{y}, t) - \alpha(\mathbf{y}, s)|}{|t - s|}.$$

Here we used (A.7) and hypothesis (2) of this lemma. Next observe that

$$\begin{aligned} |\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, t)| &= |\chi(0, t, \mathbf{x}) - \chi(0, t, \mathbf{y})| \\ &= \left| \int_0^1 \partial_\xi \chi(0, t, \mathbf{x} + \xi(\mathbf{y} - \mathbf{x})) d\xi \right| = \left| \int_0^1 \nabla_{\mathbf{x}} \chi(0, t, \mathbf{x} + \xi(\mathbf{y} - \mathbf{x})) \cdot (\mathbf{y} - \mathbf{x}) d\xi \right| \\ &\leq \int_0^1 |\nabla_{\mathbf{x}} \chi(0, t, \mathbf{x} + \xi(\mathbf{y} - \mathbf{x}))| |\mathbf{y} - \mathbf{x}| d\xi \leq 2|\mathbf{y} - \mathbf{x}|. \end{aligned}$$

Here we used Lemma A1 (1):

$$|||\nabla_{\mathbf{x}} \chi|||_{0,[0,T] \times [0,T] \times \mathbb{R}^2} \leq 2.$$

Similarly, we have

$$|||\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, t)|||_{0,[0,T] \times \mathbb{R}^2} \leq 2|t - s|.$$

Hence we have

$$|\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, s)| \leq 4|(\mathbf{x}, t) - (\mathbf{y}, s)|.$$

□

Lemma A.2. Suppose f is a scalar valued function defined on $\Lambda_s^1(T; \mathbf{v})$ satisfying

$$\sup_{0 \leq t \leq T} ||f(\cdot, t)||_{0,\gamma,\bar{\Omega}_s^1(t;\mathbf{v})} < \infty.$$

Then we have

$$\left[\left[\int_0^t f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right] \right]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq C_1(T) \left(\sup_{0 \leq t \leq T} ||f(\cdot, t)||_{0,\gamma,\bar{\Omega}_s^1(t;\mathbf{v})} \right),$$

where $[[\cdot]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}$ is the Hölder seminorm on the space-time region, and

$$C_1(T) := \left(T^{1-\gamma} + 16^\gamma T \right) = \mathcal{O}(T^{1-\gamma}).$$

If f is in $C^{0,\gamma}(\bar{\Lambda}_s^1(T; \mathbf{v}))$, then the term $\sup_{0 \leq t \leq T} ||f(\cdot, t)||_{0,\gamma,\bar{\Omega}_s^1(t;\mathbf{v})}$ can be replaced by $|||f|||_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}$, i.e.,

$$\left[\left[\int_0^t f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right] \right]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq C_1(T) |||f|||_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}.$$

Proof. Let (\mathbf{x}, t) and (\mathbf{y}, s) be two points in $\Lambda_s^1(T; \mathbf{v})$. Without loss of generality, we assume that $s \leq t$.

$$(A.11) \quad \begin{aligned} & \frac{\left| \int_0^t f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi - \int_0^s f(\chi(\xi, 0, \alpha(\mathbf{y}, s)), \xi) d\xi \right|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma} \\ & \leq \frac{\left| \int_s^t f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma} \\ & \quad + \frac{\int_0^s \left| f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) - f(\chi(\xi, 0, \alpha(\mathbf{y}, s)), \xi) \right| d\xi}{|(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma}. \end{aligned}$$

The terms on the right hand side of (A.11) can be treated as follows:

$$\begin{aligned} & \bullet \frac{\left| \int_s^t f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma} \leq (t-s)^{1-\gamma} \|f\|_{0, \bar{\Lambda}_s(T; \mathbf{v})} \leq T^{1-\gamma} \|f\|_{0, \bar{\Lambda}_s^1(T; \mathbf{v})}, \\ & \bullet \left| f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) - f(\chi(\xi, 0, \alpha(\mathbf{y}, s)), \xi) \right| \\ & \leq \left(\sup_{0 \leq \xi \leq T} [f(\cdot, \xi)]_{0, \gamma, \bar{\Omega}_s(\xi)} \right) |\chi(\xi, 0, \alpha(\mathbf{x}, t)) - \chi(\xi, 0, \alpha(\mathbf{y}, s))|^\gamma \\ & \leq \left(\sup_{0 \leq \xi \leq T} [f(\cdot, \xi)]_{0, \gamma, \bar{\Omega}_s(\xi)} \right) 2^\gamma |\alpha(\mathbf{x}, t) - \alpha(\mathbf{y}, s)|^\gamma \\ & \leq \left(\sup_{0 \leq \xi \leq T} [f(\cdot, \xi)]_{0, \gamma, \bar{\Omega}_s(\xi)} \right) 8^\gamma |(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma. \end{aligned}$$

Note that

$$\max \left\{ \|f\|_{0, \bar{\Lambda}_s^1(T; \mathbf{v})}, \left(\sup_{0 \leq t \leq T} [f(\cdot, t)]_{0, \gamma, \bar{\Omega}_s^1(t; \mathbf{v})} \right) \right\} \leq \sup_{0 \leq t \leq T} \|f(\cdot, t)\|_{0, \gamma, \bar{\Omega}_s^1(t; \mathbf{v})}.$$

Hence we have the desired result. Furthermore if f is in $C^{0, \gamma}(\bar{\Lambda}_s^1(T; \mathbf{v}))$, then the term $\sup_{0 \leq t \leq T} \|f(\cdot, t)\|_{0, \gamma, \bar{\Omega}_s^1(t; \mathbf{v})}$ can be replaced by $\|f\|_{0, \gamma, \bar{\Lambda}_s^1(T; \mathbf{v})}$, i.e.,

$$\left[\left[\int_0^t f(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right] \right]_{0, \gamma, \bar{\Lambda}_s^1(T; \mathbf{v})} \leq C_1(T) \|f\|_{0, \gamma, \bar{\Lambda}_s^1(T; \mathbf{v})}.$$

□

Lemma A.3. *Let n be the solution of (??) given by (??). Then there exists a positive constant T_1 such that n satisfies the a priori estimate:*

$$\|n\|_{0, \gamma, \bar{\Lambda}_s^1(T; \mathbf{v})} + \max_{|\alpha|=1} \|\partial^\alpha n\|_{0, \gamma, \bar{\Lambda}_s^1(T; \mathbf{v})} + \|\partial_t n\|_{0, \gamma, \bar{\Lambda}_s^1(T; \mathbf{v})} \leq R_1, \quad 0 < T \leq T_1,$$

where R_1 is a positive constant depending on K_0, δ^* and γ .

Proof. (i) Recall that n satisfies

(A.12)

$$n(\mathbf{x}, t) = n_0(\alpha(\mathbf{x}, t)) \exp \left(- \int_0^t (\nabla \cdot \mathbf{v})(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right), \quad \text{for } (\mathbf{x}, t) \in \Lambda_s^1(T; \mathbf{v}).$$

Since $\mathbf{v} \in \mathcal{B}_T$, we have

$$\|\nabla \cdot \mathbf{v}\|_{0, \gamma, \bar{\Lambda}(T)} \leq 6K_0 \delta^* \text{ in } \Lambda(T)$$

and hence (A.12) implies

$$|||n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq e^{6TK_0\delta^*} |||n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})}.$$

Furthermore if we assume T_1 is sufficiently small enough to satisfy

$$(A.13) \quad e^{6T_1K_0\delta^*} \leq 2$$

then we have

$$(A.14) \quad |||n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 2|||n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})}.$$

Next we show that $n_0(\boldsymbol{\alpha}(\mathbf{x}, t))$ is in $C^{0,\gamma}(\bar{\Lambda}_s^1(T;\mathbf{v}))$. Let (\mathbf{x}, t) and (\mathbf{y}, s) be points in $\Lambda_s^1(T;\mathbf{v})$. Without loss of generality, we assume $s \leq t$. Then we have

$$\begin{aligned} \frac{|n_0(\boldsymbol{\alpha}(\mathbf{x}, t)) - n_0(\boldsymbol{\alpha}(\mathbf{y}, s))|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma} &= \frac{|n_0(\boldsymbol{\alpha}(\mathbf{x}, t)) - n_0(\boldsymbol{\alpha}(\mathbf{y}, s))|}{|\boldsymbol{\alpha}(\mathbf{x}, t) - \boldsymbol{\alpha}(\mathbf{y}, s)|^\gamma} \left(\frac{|\boldsymbol{\alpha}(\mathbf{x}, t) - \boldsymbol{\alpha}(\mathbf{y}, s)|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|} \right)^\gamma \\ &\leq [n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})} 4^\gamma, \end{aligned}$$

and hence

$$(A.15) \quad [[n_0(\boldsymbol{\alpha})]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq [n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})} 4^\gamma.$$

On the other hand, it follows from Lemma A.2 and the fact that $\mathbf{v} \in \mathcal{B}(T)$ (see $(\mathcal{D}(2))$ in Definition 7.1) that

$$\left[\left[- \int_0^t (\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(\xi, 0, \boldsymbol{\alpha}(\mathbf{x}, t)), \xi) d\xi \right] \right]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 6C_1(T)K_0\delta^*.$$

We use (A.3) and (A.13) to get

$$(A.16) \quad \left[\left[\exp \left(- \int_0^t (\nabla \cdot \mathbf{v})(\boldsymbol{\chi}(\xi, 0, \boldsymbol{\alpha}(\mathbf{x}, t)), \xi) d\xi \right) \right] \right]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 12C_1(T)K_0\delta^*,$$

and then use (A.2), (A.13), (A.15) and (A.16) to find

$$(A.17) \quad [[n]]_{0,\gamma,\bar{\Lambda}_s^1(T)} \leq 2[n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})} 4^\gamma + 12|||n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})} C_1(T)K_0\delta^*.$$

Since $C_1(T_1) = \mathcal{O}(T_1^{1-\gamma})$, we have for T_1 sufficiently small that

$$(A.18) \quad 12C_1(T)K_0\delta^* \leq 1, \quad T \leq T_1,$$

so that (A.17) implies

$$(A.19) \quad [[n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 2[n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})} 4^\gamma + |||n_0|||_{0,\bar{\Omega}_s^1(0)}.$$

Finally combine (A.14) and (A.19) to get the desired bound

$$\begin{aligned} |||n|||_{0,\gamma,\bar{\Lambda}_s^1(T)} &\leq \max\{2^{2\gamma+1}, 3\} |||n_0|||_{0,\gamma,\bar{\Omega}_s^1(0)}, \quad 0 < T \leq T_1, \\ (A.20) \quad &\leq \max\{2^{2\gamma+1}, 3\} \delta^*. \end{aligned}$$

(ii) We now need to estimate space derivatives of n . Differentiate the continuity equation

$$\partial_t n + \sum_{i=1}^2 \partial_{x_i} (nv_i) = 0$$

with respect to x_j to find

$$(A.21) \quad \frac{D(\partial_{x_j} n)}{Dt} = - \left(\sum_{i=1}^2 \partial_{x_i x_j}^2 v_i \right) n - \left(\sum_{i=1}^2 \partial_{x_i} v_i \right) \partial_{x_j} n - \left(\sum_{i=1}^2 \partial_{x_i} n \partial_{x_j} v_i \right), \quad j = 1, 2.$$

Here $\frac{D}{Dt} = \partial_t + \mathbf{v} \cdot \nabla_x$.

Integrate (A.21) along the characteristic curve χ to obtain

$$(A.22) \quad \begin{aligned} \partial_{x_j} n(\mathbf{x}, t) &= \partial_{x_j} n_0(\boldsymbol{\alpha}(\mathbf{x}, t)) \\ &- \sum_{i=1}^2 \int_0^t \left(n \partial_{x_i x_j}^2 v_i + \partial_{x_i} v_i \partial_{x_j} n + \partial_{x_i} n \partial_{x_j} v_i \right) (\chi(\xi, 0, \boldsymbol{\alpha}(\mathbf{x}, t)), \xi) d\xi. \end{aligned}$$

Since \mathbf{v} satisfies

$$\max_{i=1,2} \left(\max_{|\alpha|=1} |||\partial^\alpha v_i|||_{0,\gamma,\bar{\Lambda}^1(T;\mathbf{v})} + \max_{|\alpha|=2} |||\partial^\alpha v_i|||_{0,\gamma,\bar{\Lambda}(T)} \right) \leq 3K_0\delta^*, \quad \text{by the definition of } \mathcal{B}(T)$$

and n satisfies

$$|||n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 2|||n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})},$$

for T sufficiently small by (A.14), we have from (A.22) that

$$(A.23) \quad \begin{aligned} &\max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \\ &\leq \max_{|\alpha|=1} |||\partial^\alpha n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})} + 6TK_0\delta^* |||n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})} + 12K_0\delta^* T \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})}. \end{aligned}$$

We assume T_1 is sufficiently small so that

$$(A.24) \quad K_0\delta^* T_1 \leq \frac{1}{24},$$

then we have from (A.24) that

$$(A.25) \quad \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 2 \max_{|\alpha|=1} |||\partial^\alpha n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})} + \frac{1}{2} |||n_0|||_{0,\bar{\Omega}_s^1(0;\mathbf{v})} \quad \text{for } T \leq T_1.$$

Next we estimate $[[\partial^\alpha n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})}$, $|\alpha| = 1$ using Lemma A.2 and (A.2).

By direct calculation, we have following estimates: For $T > 0$ sufficiently small, we have

$$(A.26) \quad \bullet \quad [[\partial_{x_j} n_0(\boldsymbol{\alpha}(\mathbf{x}, t))]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 4^\gamma \max_{|\alpha|=1} [\partial^\alpha n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})},$$

$$(A.27) \quad \bullet \quad [[\partial_{x_i x_j}^2 v_i n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 6K_0(\delta^*)^2 \max\{2^{2\gamma+1}, 3\},$$

$$(A.28) \quad \bullet \quad [[\partial_{x_i} v_i \partial_{x_j} n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 3K_0\delta^* [[\nabla n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} + 3K_0\delta^* |||\nabla n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})},$$

$$(A.29) \quad \bullet \quad [[\partial_{x_i} n \partial_{x_j} v_i]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 3K_0\delta^* \left([[\nabla n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} + \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \right).$$

We combine estimates (A.26) - (A.29) to get

$$(A.30) \quad \begin{aligned} \max_{|\alpha|=1} [[\partial^\alpha n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} &\leq 4^\gamma \max_{|\alpha|=1} [\partial^\alpha n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})} + 12K_0\delta^* C_1(T) \\ &\times \left(\delta^* \max\{2^{2\gamma+1}, 3\} + \max_{|\alpha|=1} [[\partial^\alpha n]]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} + \max_{|\alpha|=1} |||\partial^\alpha n|||_{0,\bar{\Lambda}_s^1(T;\mathbf{v})} \right). \end{aligned}$$

We assume T_1 is sufficiently small so that

$$(A.31) \quad K_0 \delta^* C_1(T_1) \leq \frac{1}{24}$$

and hence for $T \in (0, T_1]$, (A.30) implies

$$(A.32) \quad \begin{aligned} \max_{|\alpha|=1} [|\partial^\alpha n|]_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} &\leq 2^{2\gamma+1} \max_{|\alpha|=1} [\partial^\alpha n_0]_{0,\gamma,\bar{\Omega}_s^1(0;\mathbf{v})} + \delta^* \max\{24^\gamma, 3\} \\ &+ 2 \max_{|\alpha|=1} \|\partial^\alpha n_0\|_{0,\bar{\Omega}_s^1(0;\mathbf{v})} + \frac{1}{2} \|n_0\|_{0,\bar{\Omega}_s^1(0;\mathbf{v})}. \end{aligned}$$

We combine (A.25) and (A.32) to obtain

$$(A.33) \quad \max_{|\alpha|=1} \|\partial^\alpha n\|_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \leq 2\delta^* \max\{2^{2\gamma+1}, 4\}.$$

(iii) Now we estimate the time derivative of n . Recall that n satisfies

$$(A.34) \quad \partial_t n + \nabla \cdot (n\mathbf{v}) = 0.$$

Next we use (A.20), (A.33) and (A.34) to see

$$(A.35) \quad \begin{aligned} \|\partial_t n\|_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} &\leq 2 \max_{|\alpha|=1} \|\partial^\alpha n\|_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \cdot \|\mathbf{v}\|_{0,\gamma,\bar{\Lambda}(T)} + \|n\|_{0,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \|\nabla \cdot \mathbf{v}\|_{0,\gamma,\bar{\Lambda}(T)} \\ &\leq 15K_0(\delta^*)^2 \max\{2^{2\gamma+1}, 4\}. \end{aligned}$$

Finally we set

$$R_1(K_0, \delta^*, \gamma) := \max\{2^{2\gamma+1}, 3\}\delta^* + 2\delta^* \max\{2^{2\gamma+1}, 4\} + 15K_0(\delta^*)^2 \max\{2^{2\gamma+1}, 4\}$$

to see that (A.20), (A.33) and (A.35) imply the desired result. \square

A.1.2. A priori estimates for Step 2. In this part, we will give existence, uniqueness and regularity for the function ζ as given in *Step 2* of Section 7.2.

Recall from *Step 2*, ζ satisfies the exterior Neumann problem for Laplace's equation at given time $t \in [0, T]$:

$$(A.36) \quad \begin{cases} \Delta \zeta(\cdot, t) = 0, & \mathbf{x} \in \Omega_1, \\ \nabla \zeta \cdot \boldsymbol{\nu}_0 = h_0, & \mathbf{x} \in \partial\Omega_0 \end{cases} \quad \text{and} \quad \lim_{|x| \rightarrow \infty} \nabla \zeta = \mathbf{0}.$$

Lemma A.4. $h_0 \in C^{1,\gamma}(\partial\Omega_0 \times [0, T])$ and satisfies

$$\|h_0\|_{1,\gamma,\partial\Omega_0 \times [0,T]} \leq \delta^* + 6\bar{C}_0 K_0 R_1(\delta^*)^2,$$

where \bar{C}_0 is a positive constant.

Proof. Recall that $h_0 = \partial_t g - (n\mathbf{v}) \cdot \boldsymbol{\nu}_0$. Then

$$(A.37) \quad \begin{aligned} \|h_0\|_{1,\gamma,\partial\Omega_0 \times [0,T]} &\leq \|\partial_t g\|_{1,\gamma,\partial\Omega_0 \times [0,\infty)} \\ &+ \|nv_1\nu_{01}\|_{1,\gamma,\partial\Omega_0 \times [0,T]} + \|nv_2\nu_{02}\|_{1,\gamma,\partial\Omega_0 \times [0,T]}. \end{aligned}$$

Since the product of Hölder continuous functions is again Hölder continuous (see [34], pg. 53), we have

$$\begin{aligned}
& |||nv_1\nu_{01}|||_{1,\gamma,\partial\Omega_0\times[0,T]} + |||nv_2\nu_{02}|||_{1,\gamma,\partial\Omega_0\times[0,T]} \\
& \leq \bar{C}_0 \left(|||n|||_{1,\gamma,\partial\Omega_0\times[0,T]} |||v_1|||_{1,\gamma,\partial\Omega_0\times[0,T]} |||\nu_{01}|||_{1,\gamma,\partial\Omega_0\times[0,T]} \right. \\
& \quad \left. + |||n|||_{1,\gamma,\partial\Omega_0\times[0,T]} |||v_2|||_{1,\gamma,\partial\Omega_0\times[0,T]} |||\nu_{02}|||_{1,\gamma,\partial\Omega_0\times[0,T]} \right) \\
& \leq 6\bar{C}_0 K_0 R_1 (\delta^*)^2.
\end{aligned}
\tag{A.38}$$

Here \bar{C}_0 is a positive constant and we used Lemma A.3, $\max_{i=1,2} |||v_i|||_{1,\gamma,\bar{\Lambda}(T)} \leq 3K_0\delta^*$, and inequalities

$$|||n|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq |||n|||_{1,\gamma,\bar{\Lambda}_s^1(T;\mathbf{v})} \quad \text{and} \quad |||v_i|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq |||v_i|||_{1,\gamma,\bar{\Lambda}(T)}.$$

Hence in (A.37), we use (A.38) and the assumption (A2) of Section 7.2 to get

$$|||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} \leq \delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2.$$

□

Recall the annulus region (??) in Section 7.2:

$$\Omega_* = \{\mathbf{x} \in \mathbb{R}^2 : \frac{\delta_* 2}{2} < |\mathbf{x}| < 2\delta^*\}.$$

The following existence and uniqueness result of two-dimensional exterior Neumann problem (A.36) is due to the results of Bers [11] and Finn-Gilbarg [30].

Lemma A.5. *Suppose the boundary data h_0 is in $C^{1,\gamma}(\partial\Omega_0 \times [0, \infty))$ as provided by Lemma A.4. Then there exists a unique solution ζ up to constant of (A.36) satisfying the following estimates: For the compactly supported subset Ω_* of Ω_1 , we have*

(1)

$$|||\nabla\zeta|||_{1,\gamma,\bar{\Omega}_*\times[0,T]} + \sup_{0 \leq t \leq T} \max_{|\alpha|=3} |||\partial^\alpha \zeta(\cdot, t)|||_{0,\gamma,\bar{\Omega}_*} \leq \bar{R}_1, \quad 0 \leq t \leq T,$$

where \bar{R}_1 is a positive constant which depends only on Ω_0 , δ^* .

(2) Let $h_0^{(n)} \in C^{1,\tau}(\partial\Omega_0 \times [0, \infty))$ be a sequence of boundary data satisfying the bound (A.39) and

$$h_0^{(n)} \rightarrow h_0 \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0, \infty)) \quad \text{as } n \rightarrow \infty.$$

Suppose $\nabla\zeta^{(n)}$ and $\nabla\zeta$ are the corresponding solutions to the above exterior Neumann problem (A.36) for data $h_0^{(n)}, h_0$ respectively. Then we have

$$\nabla\zeta^{(n)} \rightarrow \nabla\zeta \quad \text{in } (C^{1,\tau}(\bar{\Omega}_* \times [0, T]))^2 \quad \text{as } n \rightarrow \infty.$$

Proof. (i) Since Ω_* is compactly supported in Ω_1 , it follows from the interior Schauder estimates ([34], Section 6.1), we have

$$\begin{aligned}
& |||\nabla\zeta(\cdot, t)|||_{1,\gamma,\bar{\Omega}_*} + \max_{|\alpha|=3} |||\partial^\alpha \zeta(\cdot, t)|||_{0,\gamma,\bar{\Omega}_*} \\
& \leq C_0(\Omega_0, \Omega_*) |||h_0|||_{1,\gamma,\partial\Omega_0\times[0,T]} \\
& \leq C_0(\Omega_0, \Omega_*) (\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2), \quad 0 \leq t \leq T.
\end{aligned}
\tag{A.40}$$

Let $0 \leq s < t$. Then it follows from Laplace's equation and the boundary condition that

$$\begin{cases} \Delta \left(\frac{\zeta(\mathbf{x}, t) - \zeta(\mathbf{x}, s)}{|t-s|^\gamma} \right) = 0, & \mathbf{x} \in \Omega_1, \quad 0 \leq s < t \leq T, \\ \nabla \left(\frac{\zeta(\mathbf{x}, t) - \zeta(\mathbf{x}, s)}{|t-s|^\gamma} \right) \cdot \boldsymbol{\nu}_0 = \frac{h_0(x, t) - h_0(\mathbf{x}, s)}{|t-s|^\gamma}, & \mathbf{x} \in \partial\Omega_0. \end{cases}$$

We now apply the global Schauder estimate ([34], Section 6.2) to get

$$\begin{aligned} & \frac{|\nabla \zeta(\mathbf{x}, t) - \nabla \zeta(\mathbf{x}, s)|}{|t-s|^\gamma} \\ & \leq C_1(\Omega_0, \Omega_*) |||h_0|||_{1, \gamma, \partial\Omega_0 \times [0, T]} \\ (A.41) \quad & \leq C_1(\Omega_0, \Omega_*)(\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2), \quad \mathbf{x} \in \Omega_1. \end{aligned}$$

We take the supremum over $t \neq s$ to get

$$(A.42) \quad \sup_{t \neq s} \frac{|\nabla \zeta(\mathbf{x}, t) - \nabla \zeta(\mathbf{x}, s)|}{|t-s|^\gamma} \leq C_1(\Omega_0, \Omega_*)(\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2), \quad \mathbf{x} \in \Omega_1.$$

Let $(\mathbf{x}, t) \neq (\mathbf{y}, s)$ and without loss of generality, assume that $\mathbf{x} \neq \mathbf{y}, t \neq s$. From (A.40) and (A.43), the Hölder quotient satisfies

$$\begin{aligned} \frac{|\nabla \zeta(\mathbf{x}, t) - \nabla \zeta(\mathbf{y}, s)|}{|(\mathbf{x}, t) - (\mathbf{y}, s)|^\gamma} & \leq \frac{|\nabla \zeta(\mathbf{x}, t) - \nabla \zeta(\mathbf{x}, s)|}{|t-s|^\gamma} + \frac{|\nabla \zeta(\mathbf{x}, s) - \nabla \zeta(\mathbf{y}, s)|}{|\mathbf{x} - \mathbf{y}|^\gamma} \\ & \leq [\nabla \zeta(\mathbf{x}, \cdot)]_{0, \gamma, [0, T]} + [\nabla \zeta(\cdot, s)]_{0, \gamma, \Omega_s(s)} \\ & \leq (C_0(\Omega_0, \Omega_*) + C_1(\Omega_0, \Omega_*))(\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2). \end{aligned}$$

Taking sup over the time-space region $\Omega_* \times [0, T]$, we have

$$(A.43) \quad [\nabla \zeta]_{0, \gamma, \bar{\Omega}_* \times [0, T]} \leq (C_0(\Omega_0, \Omega_*) + C_1(\Omega_0, \Omega_*))(\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2).$$

Similarly we can estimate $\max_{|\alpha|=2} |||\partial^\alpha \zeta|||_{0, \gamma, \bar{\Omega}_* \times [0, T]}$ to get

$$(A.44) \quad \max_{|\alpha|=2} |||\partial^\alpha \zeta|||_{0, \gamma, \bar{\Omega}_* \times [0, T]} \leq C_2(\Omega_0, \Omega_*)(\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2).$$

Finally we combine (A.40), (A.43) and (A.44) to get

$$|||\nabla \zeta|||_{1, \gamma, \bar{\Omega}_* \times [0, T]} + \sup_{0 \leq t \leq T} \max_{|\alpha|=3} |||\partial^\alpha \zeta(\cdot, t)|||_{0, \gamma, \bar{\Omega}_*} \leq \bar{R}_1,$$

where $\bar{R}_1 := (C_0(\Omega_0, \Omega_*) + C_1(\Omega_0, \Omega_*) + C_2(\Omega_0, \Omega_*))(\delta^* + 6\bar{C}_0 K_0 R_1 (\delta^*)^2)$.

(ii) The difference $\zeta^{(n)} - \zeta$ satisfies

$$\begin{cases} \Delta(\zeta^{(n)}(\cdot, t) - \zeta(\cdot, t)) = 0, & \mathbf{x} \in \Omega_1, \\ \nabla(\zeta^{(n)} - \zeta) \cdot \boldsymbol{\nu}_0 = h_0^{(n)} - h_0, & \mathbf{x} \in \partial\Omega_0 \quad \text{and} \quad \zeta^{(n)} - \zeta = 0 \quad \text{on } \partial B(0, 3\delta^*). \end{cases}$$

By the Schauder estimates (Section 6.2 in [34]), we have

$$|||\nabla \zeta^{(n)}(\cdot, t) - \nabla \zeta(\cdot, t)|||_{1, \tau, \bar{\Omega}_1} \leq C |||h_0^{(n)} - h_0|||_{1, \tau, \partial\Omega_0}.$$

Letting $n \rightarrow \infty$, it follows from the above inequality and hypothesis (2) of this lemma that

$$(A.45) \quad \nabla \zeta^{(n)}(\cdot, t) \rightarrow \nabla \zeta(\cdot, t) \quad \text{in } (C^{1, \tau}(\bar{\Omega}_1))^2, \quad 0 \leq t \leq T.$$

For the time-estimates we apply the same method as in (i) to get

$$(A.46) \quad \nabla \zeta^{(n)}(\mathbf{x}, \cdot) \rightarrow \nabla \zeta(\mathbf{x}, \cdot) \quad \text{in } (C^{1, \tau}([0, T]))^2, \quad \mathbf{x} \in \bar{\Omega}_*.$$

We combine (A.45) and (A.46) to see

$$\nabla \zeta^{(n)} \rightarrow \nabla \zeta \quad \text{in } (C^{1,\tau}(\bar{\Omega}_1 \times [0, T]))^2.$$

This yields

$$\nabla \zeta^{(n)} \rightarrow \nabla \zeta \quad \text{in } (C^{1,\tau}(\bar{\Omega}_* \times [0, T]))^2.$$

□

A.1.3. A priori estimates for Step 3. We now need to give existence, uniqueness and regularity for the interface of Step 3 of Section 7.2.

Lemma A.6. *1. Assume that ζ satisfies estimate (1) of Lemma A.5:*

$$\|\nabla \zeta\|_{1,\gamma,\bar{\Omega}_* \times [0,T]} + \sup_{0 \leq t \leq T} \max_{|\alpha|=3} \|\partial^\alpha \zeta(\cdot, t)\|_{0,\gamma,\bar{\Omega}_*} \leq \bar{R}_1.$$

Then there exists a unique solution for the interface system (??) satisfying

$$(\theta, n_s, r) \in (C^{1,\gamma}(\mathbb{R} \times [0, T]))^3 \quad \text{and} \quad (\partial_\beta \theta, \partial_\beta n_s, \partial_\beta r)(\cdot, t) \in (C^{2,\gamma}(\mathbb{R}))^3, \quad t \in [0, T].$$

Moreover, we have

$$\begin{aligned} & \bullet \|\theta\|_{1,\gamma,\mathbb{R} \times [0,T]} + \|n_s\|_{1,\gamma,\mathbb{R} \times [0,T]} + \|r\|_{1,\gamma,\mathbb{R} \times [0,T]} \\ & \leq 2 \left(\|\theta_0\|_{1,\gamma,\mathbb{R}} + \|n_{s0}\|_{1,\gamma,\mathbb{R}} + \|r_0\|_{1,\gamma,\mathbb{R}} \right), \\ & \bullet \|\partial_\beta \theta(\cdot, t)\|_{2,\gamma,\mathbb{R}} + \|\partial_\beta n_s(\cdot, t)\|_{2,\gamma,\mathbb{R}} + \|\partial_\beta r(\cdot, t)\|_{2,\gamma,\mathbb{R}} \\ & \leq 2 \left(\|\partial_\beta \theta_0\|_{2,\gamma,\mathbb{R}} + \|\partial_\beta n_{s0}\|_{2,\gamma,\mathbb{R}} + \|\partial_\beta r_0\|_{2,\gamma,\mathbb{R}} \right). \end{aligned}$$

2. Let

$$\nabla \zeta^{(n)} \rightarrow \nabla \zeta \quad \text{in } (C^{1,\tau}(\bar{\Omega}_* \times [0, T]))^2 \quad \text{as given by (2) of Lemma A.4,}$$

and let $(\theta^{(n)}, n_s^{(n)}, r^{(n)})$ and (θ, n_s, r) be the solutions of the sheath system corresponding to $\nabla \zeta^{(n)}$ and $\nabla \zeta$ respectively. Then we have

$$(\theta^{(n)}, n_s^{(n)}, r^{(n)}) \rightarrow (\theta, n_s, r) \quad \text{in } (C^{1,\tau}(\mathbb{R} \times [0, T]))^3 \quad \text{as } n \rightarrow \infty.$$

Proof. The result is just continuity with respect to data for the hyperbolic system (??). The proof of convergence in $C^1(\mathbb{R} \times [0, T])$ follows from the argument in [25]. The proof of Hölder norms $C^{1,\gamma}(\mathbb{R} \times [0, T])$ is similar to that of [25] (see [40]). □

A.1.4. A priori estimates for Step 4. We next present the existence, uniqueness and regularity of the ion density n in the region $\Lambda_s^2(T)$ given by Step 4 of Section 7.2.

Lemma A.7. *Let n be the ion-density obtained from Step 4. Then the formulas in Step 4, namely n satisfies the differential equations:*

$$\begin{cases} \frac{d}{ds} \chi(s, t_0, \alpha) = \mathbf{v}(\chi(s, t_0, \alpha), s), & s > t_0, \\ \frac{d}{ds} \ln n(\chi(s, t_0, \alpha), s) = -(\nabla \cdot \mathbf{v})(\chi(s, t_0, \alpha), s), \end{cases}$$

subject to initial and boundary data:

$$\chi(t_0, t_0, \alpha) = \alpha \quad \text{and} \quad n(\alpha, t_0) = \begin{cases} n_0(\alpha) & t_0 = 0, \\ n_s(\alpha, t_0) & t_0 > 0, \end{cases}$$

are indeed valid. Furthermore for sufficiently small T , the following estimates hold:

$$n \in C^{1,\gamma}(\Lambda_s^2(T; \mathbf{v})) \quad \text{and} \quad \|n\|_{1,\gamma;\Lambda_s^2(T;\mathbf{v})} \leq R_2,$$

where R_2 is a positive constant depending only on K_0, δ^*, γ .

Proof. The proof follows from Remark A.1, i.e. since backward characteristics starting at a point $(\mathbf{x}, t) \in \Lambda_s^2(T; \mathbf{v})$ can be traced back to a point $(\boldsymbol{\alpha}, 0)$ in the absence of the sheath interface, the presence of the sheath interface means backward characteristics must hit either a point in $\Omega_s^2(0; \mathbf{v})$ or a point in the sheath interface. Furthermore, the segment of backwards characteristic between (\mathbf{x}, t) and $(\boldsymbol{\alpha}, 0)$ can hit the sheath interface at most once. Indeed, the backwards characteristic can enter but not exit the domain $\Lambda_s(T; \mathbf{v}) = \Lambda_s^1(T; \mathbf{v}) \cap \Lambda_s^2(T; \mathbf{v})$ through the interface surface at time $0 < t < T$. This is because initially $\mathbf{u}_0 = -\boldsymbol{\nu}$ on $\mathcal{S}(0)$, hence $|\mathbf{v} \cdot \boldsymbol{\nu} + 1| < \varepsilon$ on $\mathcal{S}(t)$ for $0 < t < T$ by (A1) and Theorem ??, where $T > 0$ is sufficiently small depending only on initial data, boundary data, and $\varepsilon > 0$. By choosing $\varepsilon > 0$ sufficiently small, the vector field for the characteristic $\frac{d\boldsymbol{\chi}}{dt} = \mathbf{v}(\boldsymbol{\chi}, t)$ always points into the domain $\Lambda_s(T; \mathbf{v})$ at the point of intersection with the sheath interface.

Hence the formulas follow from (??). Furthermore the regularity estimates in the statement of the lemma can be obtained in a similar manner as in Lemma A.3. \square

We combine Lemma A.3 and Lemma A.7 to get the regularity result for n in the sheath region.

Lemma A.8. *For sufficiently small T , we have*

$$n \in C^{1,\gamma}(\bar{\Lambda}_s(T; \mathbf{v})) \quad \text{and} \quad \|n\|_{1,\gamma;\bar{\Lambda}_s(T;\mathbf{v})} \leq R_1 + R_2.$$

A.1.5. A priori estimates for Step 5. We next give the existence, uniqueness and regularity for the function ϕ defined in Step 5.

Consider Poisson's equation on the space-time sheath region $\Lambda_s(T; \mathbf{v})$: Let $t \in [0, T]$ be given and ϕ satisfy

$$(A.47) \quad \begin{cases} \Delta \phi = n & \text{in } \Omega_s(t; \mathbf{v}), \\ \nabla \phi \cdot \boldsymbol{\nu}_0 = g & \text{on } \partial\Omega_0 \quad \text{and} \quad \phi = -\ln n_s & \text{on } \mathcal{S}(t). \end{cases}$$

Lemma A.9. *Let n be an ion density in the sheath region $\Lambda_s(T; \mathbf{v})$ and satisfy the a priori estimate in Lemma A.8. Then Poisson's equation (A.47) has a unique solution ϕ satisfying the following estimate:*

$$\max_{1 \leq |\alpha| \leq 2} \|\partial^\alpha \phi\|_{0,\gamma;\bar{\Lambda}_s(T;\mathbf{v})} + \sup_{0 \leq t \leq T} \max_{|\alpha|=3} \|\partial^\alpha \phi(\cdot, t)\|_{0,\gamma;\bar{\Omega}_s(t;\mathbf{v})} \leq R_3.$$

Here R_3 is a positive constant only depending on $K_0, \delta_{*i}, i = 1, 2$ and δ^* respectively.

Proof. (i) Differentiation of (A.47) with respect to t shows that $\partial_t \phi$ satisfies the mixed Dirichlet-Neumann problem for Poisson's equation.

$$\begin{cases} \Delta \partial_t \phi = -\operatorname{div}(n \mathbf{v}) & \text{in } \Omega_s(t; \mathbf{v}), \\ \nabla \partial_t \phi \cdot \boldsymbol{\nu}_0 = \partial_t g & \text{on } \partial\Omega_0 \quad \text{and} \quad \partial_t \phi = -\partial_t \ln n_s & \text{on } \mathcal{S}(t), \end{cases}$$

where we used $\nabla \phi \cdot \boldsymbol{\nu} = 0$ and $\nabla n \cdot \boldsymbol{\nu} = 0$ on the interface $\mathcal{S}(t)$, $0 \leq t \leq T$.

By the direct application of Hölder estimates of the first derivatives given in ([34], Section 8), we have

$$\|\partial_t \phi\|_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq \bar{C}_1 \left(\|\partial_t g\|_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + \|n\mathbf{v}\|_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + \|\partial_t \ln n_s\|_{2,\gamma,\mathbb{R}} \right).$$

Here \bar{C}_1 depends on Ω_0 and $\mathcal{S}(t)$, but we can choose uniform \bar{C}_1 independent of t and depending only on δ_* and δ^* for sufficiently small T , $0 \leq t \leq T$.

On the other hand, since $\mathbf{v} \in \mathcal{B}(T)$, we have

$$\begin{aligned} \bullet \quad \|n\mathbf{v}\|_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} &\leq \bar{C}_2 \|n\|_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \|\mathbf{v}\|_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq \bar{C}_3 K_0 (R_2 + R_3) \delta^*, \\ \bullet \quad \|\partial_t g\|_{1,\gamma,\bar{\Omega}_s(t;\mathbf{v})} &\leq \delta^* \quad \text{by the assumption (A2) in Section 7.2,} \end{aligned}$$

where \bar{C}_2 and \bar{C}_3 are some positive constants.

It follows from the interface equation (??) that

$$\begin{aligned} \frac{\partial_t n_s}{n_s} &= - \left(\frac{4 \sin \beta \sin \theta \theta}{r} \right) \partial_\beta \theta - \left(\frac{2 \sin \beta \tilde{V} \cos \theta}{r n_s} \right) \partial_\beta n_s, \\ \tilde{V} &= -1 - \frac{\nabla \zeta \cdot (\cos \theta, \sin \theta)}{n_s}. \end{aligned}$$

We use the above relation and the estimates from Lemma A.5 (1) to obtain

$$\|\partial_t \ln n_s\|_{2,\gamma,\mathbb{R}} \leq C(\delta_{*1}, \delta^*).$$

Here $C(\delta_{*1}, \delta^*)$ is a positive constant depending only on δ_{*1}, δ^* . Hence we have

$$(A.48) \quad \sup_{0 \leq t \leq T} \|\partial_t \phi(\cdot, t)\|_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} \leq R_{3,0}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{for } t \in [0, T].$$

(ii) It follows from the Schauder estimates (Section 6.2 in [34]) that

$$\begin{aligned} \|\phi(\cdot, t)\|_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})} &\leq \bar{C}_4 \left(\|n(\cdot, t)\|_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})} + \|g(\cdot, t)\|_{1,\gamma,\partial\Omega_0} + \|\ln n_s(\cdot, t)\|_{2,\gamma,\mathbb{R}} \right) \\ (A.49) \quad &\leq R_{3,1}(K_0, \delta_{*1}, \delta_{*2}, \delta^*), \quad 0 \leq t \leq T. \end{aligned}$$

Here \bar{C}_4 depends only on the Ω_0 and $\mathcal{S}(t)$, but again we can choose \bar{C}_4 depending only on δ_* and δ^* for sufficiently small T , $0 \leq t \leq T$.

Let (\mathbf{x}, t) and (\mathbf{y}, s) be any points in $\Lambda_s(T; \mathbf{v})$. Without loss of generality, we assume that $0 \leq s < t$. By assumption (A4) of Section 7.2, we have a contracting interface so that

$$\Omega_s(t; \mathbf{v}) \subset \Omega_s(s; \mathbf{v}), \quad 0 \leq s < t \leq T \ll 1.$$

Hence $\mathbf{x} \in \Omega_s(t; \mathbf{v})$. Then inequality (A.48) implies, for $\mathbf{x} \in \Omega_s(t; \mathbf{v})$

$$(A.50) \quad \max_{1 \leq |\alpha| \leq 2} \|\partial^\alpha \phi(\mathbf{x}, \cdot)\|_{0,\gamma,[0,T]} \leq R_{3,0}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) T^{1-\gamma}.$$

We combine (A.49) and (A.50) and choose T sufficiently small to get

$$(A.51) \quad \max_{1 \leq |\alpha| \leq 2} \|\partial^\alpha \phi\|_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq R_{3,2}(K_0, \delta_{*1}, \delta_{*2}, \delta^*).$$

(In fact the above argument holds for the expanding interfaces as well).

(iii) On the other hand, $\partial_{x_i} \phi, i = 1, 2$ satisfies

$$(A.52) \quad \begin{cases} \Delta \partial_{x_i} \phi = \partial_{x_i} n & \text{in } \Omega_s(t; \mathbf{v}), \\ \nabla \partial_{x_i} \phi \cdot \boldsymbol{\nu}_0 = \partial_{x_i} g & \text{on } \partial\Omega_0 \quad \text{and} \quad \partial_{x_i} \phi = -\partial_{x_i} \ln n_s \quad \text{on } \mathcal{S}(t). \end{cases}$$

Again, it follows from the Poisson equation and the Schauder estimates (Section 6.2 in [34]) that

$$(A.53) \quad \begin{aligned} & \|\partial_{x_i} \phi(\cdot, t)\|_{2, \gamma, \bar{\Omega}_s(t; \mathbf{v})} \\ & \leq \bar{C}_5 \left(\|\partial_{x_i} n(\cdot, t)\|_{0, \gamma, \bar{\Omega}_s(t; \mathbf{v})} + \|\partial_{x_i} g(\cdot, t)\|_{1, \gamma, \partial \Omega_0} + \|\partial_{x_i} \ln n_s(\cdot, t)\|_{2, \gamma, \mathbb{R}} \right). \end{aligned}$$

Here \bar{C}_5 depends on $\mathcal{S}(t)$, but we can choose \bar{C}_5 depending only on $\delta_{*i}, i = 1, 2$ and δ^* for sufficiently small $T, 0 \leq t \leq T$.

The first two terms in the right hand side of (A.53) can be bounded by a quantity depending on δ^* using Lemma A.7 and assumptions (A1)-(A2) of the boundary data in Section 7.2, i.e.,

$$(A.54) \quad \|\partial_{x_i} n(\cdot, t)\|_{0, \gamma, \bar{\Omega}_s(t; \mathbf{v})} + \|\partial_{x_i} g(\cdot, t)\|_{1, \gamma, \partial \Omega_0} \leq \bar{C}_6.$$

Here \bar{C}_6 is a positive constant depending only on δ_{*1} and δ^* .

Now we estimate the third term $\|\partial_{x_i} \ln n_s\|_{2, \gamma, \mathbb{R}}$ as follows. It follows from (??) that we have

$$\partial_{x_1} = -\frac{2 \sin \beta}{r} \partial_\beta$$

and similarly we can express ∂_{x_2} in terms of ∂_β . Therefore we have

$$(A.55) \quad \|\partial_{x_i} \ln n_s(\cdot, t)\|_{2, \gamma, \mathbb{R}} = \left\| \frac{\partial_{x_i} n_s(\cdot, t)}{n_s(\cdot, t)} \right\|_{2, \gamma, \mathbb{R}} \leq \bar{C}_7 \quad i = 1, 2.$$

Here \bar{C}_7 is a positive constant depending only on δ_{*1} and δ^* .

Combining estimates (A.54) and (A.55), we obtain

$$\max_{|\alpha|=1} \|\partial^\alpha \phi(\cdot, t)\|_{2, \gamma, \bar{\Omega}_s(t; \mathbf{v})} \leq R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{for } t \in [0, T].$$

The above inequality implies

$$\sup_{0 \leq t \leq T} \max_{|\alpha|=1} \|\partial^\alpha \phi(\cdot, t)\|_{0, \gamma, \bar{\Omega}_s(t; \mathbf{v})} \leq R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{for } t \in [0, T].$$

In particular we have

$$(A.56) \quad \sup_{0 \leq t \leq T} \max_{|\alpha|=3} \|\partial^\alpha \phi(\cdot, t)\|_{0, \gamma, \bar{\Omega}_s(t; \mathbf{v})} \leq R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) \quad \text{for } t \in [0, T].$$

We set $R_3(K_0, \delta_{*1}, \delta_{*2}, \delta^*) := R_{3,2}(K_0, \delta_{*1}, \delta_{*2}, \delta^*) + R_{3,3}(K_0, \delta_{*1}, \delta_{*2}, \delta^*)$ and use (A.53) and (A.56) to get the desired result. \square

A.1.6. A priori estimates for Step 6. In this part, we give the existence, uniqueness and regularity for the ion velocity $\hat{\mathbf{u}}$ defined in Step 6 of Section 7.2.

Consider the Burgers' equation with a known source $\nabla \phi$:

$$(A.57) \quad \partial_t \hat{\mathbf{u}} + (\hat{\mathbf{u}} \cdot \nabla) \hat{\mathbf{u}} = \nabla \phi, \quad (\mathbf{x}, t) \in \mathbb{R}^2 \times \mathbb{R}_+.$$

Lemma A.10. *Suppose the source $\nabla \phi$ satisfies the estimates obtained in Lemma A.9. Also assume initial data \mathbf{u}_0 satisfy the assumption (A3) of Section 7.2 so that*

- (1) $\nabla \phi \in C^{1, \gamma}(\bar{\Lambda}_s(T; \mathbf{v}))$ and $\nabla \phi(\cdot, t) \in C^{2, \gamma}(\Omega_s(t; \mathbf{v}))$;
- (2) for each $\alpha \in \mathbb{R}^2$, the real parts of the eigenvalues of $\nabla \mathbf{u}_0(\alpha)$ are non-negative;

there is a positive constant T_2 such that (A.57) has a unique solution $\hat{\mathbf{u}} \in C^{1,\gamma}(\bar{\Lambda}_s(T; \mathbf{v}))$ satisfying

$$(A.58) \quad \begin{aligned} \det \Gamma(\boldsymbol{\alpha}, t) &> 0 \quad \text{and} \\ \hat{\mathbf{u}}(\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha})) &= \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \nabla \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}), s) ds, \quad t \in [0, T_2], \\ \hat{\mathbf{u}}(\mathbf{x}, t) \cdot \mathbf{x} &\leq -\frac{\eta_0}{2} |\mathbf{x}|^2, \quad (\mathbf{x}, t) \in (B(0, r_b + 6K_0 \delta^* T_2) - \Omega_0) \times [0, T_2], \end{aligned}$$

where T_2 is a positive constant and

$$\frac{d\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha})}{dt} = \hat{\mathbf{u}}(\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha}), t) \quad \text{and} \quad \Gamma(\boldsymbol{\alpha}, t) = \nabla \hat{\mathbf{u}}(\boldsymbol{\alpha}, t).$$

Proof. (i) Along the particle path $\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha})$, system (A.57) becomes

$$(A.59) \quad \frac{D\hat{\mathbf{u}}}{Dt} = \nabla \phi, \quad \text{where} \quad \frac{D}{Dt} = \partial_t + \hat{\mathbf{u}} \cdot \nabla$$

Any smooth solution of (A.58) will satisfy

$$(A.60) \quad \frac{d^2 \boldsymbol{\chi}(t, 0, \boldsymbol{\alpha})}{dt^2} = \nabla \phi(\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha}), t); \quad \boldsymbol{\chi}(\boldsymbol{\alpha}, 0) = \boldsymbol{\alpha}, \quad \frac{d\boldsymbol{\chi}(0, 0, \boldsymbol{\alpha})}{dt} = \mathbf{u}_0(\boldsymbol{\alpha}).$$

Since $\nabla \phi(\cdot, t)$ is Lipschitz continuous and uniformly bounded, there exists a unique characteristic curve $\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha})$ satisfying (A.60) locally in time t . Now we integrate (A.60) to get

$$(A.61) \quad \frac{d\boldsymbol{\chi}(t, 0, \boldsymbol{\alpha})}{dt} = \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \nabla \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}), s) ds.$$

and integration of the above equation yields

$$(A.62) \quad \begin{aligned} \boldsymbol{\chi}(t, 0, \boldsymbol{\alpha}) &= \boldsymbol{\chi}(0, 0, \boldsymbol{\alpha}) + t\mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \int_0^{t_1} \nabla \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}), s) ds dt_1 \\ &= \boldsymbol{\alpha} + t\mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \int_0^{t_1} \nabla \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}), s) ds dt_1. \end{aligned}$$

Next we differentiate (A.62) with respect to $\boldsymbol{\alpha}$ to get

$$(A.63) \quad \Gamma(\boldsymbol{\alpha}, t) = I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + \int_0^t \int_0^{t_1} (\nabla \otimes \nabla) \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}), s) \Gamma(\boldsymbol{\alpha}, s) ds dt_1.$$

Set

$$y(t) = |\Gamma(\boldsymbol{\alpha}, t) - I - t\nabla \mathbf{u}_0(\boldsymbol{\alpha})|,$$

where $|\cdot|$ denotes any norm on 2×2 matrices so that we have

$$y(t) \leq \int_0^t \int_0^{t_1} |(\nabla \otimes \nabla) \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}), s)| |\Gamma(\boldsymbol{\alpha}, s)| ds dt_1.$$

Since

$$\begin{aligned} |\Gamma(\boldsymbol{\alpha}, t)| &= |\Gamma(\boldsymbol{\alpha}, t) - I - t\nabla \mathbf{u}_0(\boldsymbol{\alpha})| + |I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha})| \\ &\leq y(t) + 1 + d_1 t, \end{aligned}$$

where $\sup_{\boldsymbol{\alpha} \in \Omega(0)} |\nabla \mathbf{u}_0(\boldsymbol{\alpha})| \leq d_1$, we have from (A.63) that

$$y(t) \leq \int_0^t \int_0^{t_1} d_2(y(s) + 1 + d_1 s) ds dt_1,$$

where $|\nabla \otimes \nabla \phi(\boldsymbol{\chi}(s, 0, \boldsymbol{\alpha}))| \leq d_2$ for $\boldsymbol{\alpha} \in \Omega(0), 0 \leq s \leq T$. Hence we have

$$y(t) \leq d_2 \left(\frac{t^2}{2} + d_1 \frac{t^3}{3} \right) + d_2 \int_0^t \int_0^{t_1} y(s) ds dt_1$$

and by Appendix B

$$y(t) \leq d_3 t^2 \quad \text{on } 0 \leq t \leq T,$$

for sufficiently small T , i.e.,

$$|\Gamma(\boldsymbol{\alpha}, t) - I - t \nabla \mathbf{u}_0(\boldsymbol{\alpha})| \leq d_3 t^2, \quad 0 \leq t \leq T.$$

Define

$$tD(t, \boldsymbol{\alpha}) := \Gamma(\boldsymbol{\alpha}, t) - I - t \nabla \mathbf{u}_0(\boldsymbol{\alpha}).$$

Then we have

$$|D(t, \boldsymbol{\alpha})| \leq d_3 t \quad \text{for some constant } d_3 > 0$$

and

$$\det(\Gamma(\boldsymbol{\alpha}, t)) = \det(I + t(\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + D(t, \boldsymbol{\alpha}))).$$

Let $\lambda_i(\boldsymbol{\alpha}, t)$ and $\lambda_i(\boldsymbol{\alpha}), i = 1, 2$ be the eigenvalues of a matrix $\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + D(t, \boldsymbol{\alpha})$ and $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$ respectively. Then we can see that

$$\lambda_i(\boldsymbol{\alpha}, t) = \lambda_i(\boldsymbol{\alpha}) + \mathcal{O}(t).$$

By the Cayley-Hamilton theorem, we have

$$\begin{aligned} \det(\Gamma(\boldsymbol{\alpha}, t)) &= (t\lambda_1(\boldsymbol{\alpha}, t) + 1)(t\lambda_2(\boldsymbol{\alpha}, t) + 1) \\ &= (t\lambda_1(\boldsymbol{\alpha}) + 1)(t\lambda_2(\boldsymbol{\alpha}) + 1) + \mathcal{O}(t^2) \\ &= \det(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha})) + \mathcal{O}(t^2). \end{aligned}$$

As long as $0 \leq t \leq T \ll 1$, the sign of $\det(\Gamma(\boldsymbol{\alpha}, t))$ will be determined by $\det(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha}))$.

Next we calculate $\det(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha}))$. Let us set the characteristic polynomial of $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$ by $P(\boldsymbol{\alpha}, \lambda)$. Then we have

$$P(\boldsymbol{\alpha}, \lambda) \equiv \det(\nabla \mathbf{u}_0(\boldsymbol{\alpha}) - \lambda I) = (\lambda_1(\boldsymbol{\alpha}) - \lambda)(\lambda_2(\boldsymbol{\alpha}) - \lambda),$$

where $\lambda_i(\boldsymbol{\alpha})$ are the eigenvalues of $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$. Hence

$$\begin{aligned} \det(I + t\nabla \mathbf{u}_0(\boldsymbol{\alpha})) &= t^2 \det(\nabla \mathbf{u}_0(\boldsymbol{\alpha}) + t^{-1}I) = t^2 P(\boldsymbol{\alpha}, -t^{-1}) \\ &= t^2 (\lambda_1(\boldsymbol{\alpha}) + t^{-1})(\lambda_2(\boldsymbol{\alpha}) + t^{-1}) \\ &= (t\lambda_1(\boldsymbol{\alpha}) + 1)(t\lambda_2(\boldsymbol{\alpha}) + 1). \end{aligned}$$

Since by assumption (2) above, real parts of the eigenvalues of the Jacobian matrix $\nabla \mathbf{u}_0(\boldsymbol{\alpha})$ are nonnegative, we have

$$\det \Gamma(\boldsymbol{\alpha}, t) \geq \Pi_{q=1}^2 [1 + t \operatorname{Re} \lambda_q(\boldsymbol{\alpha})] + \mathcal{O}(t^2) > 0, \quad 0 \leq t \leq T \ll 1.$$

Hence the Lagrangian map is a C^1 -diffeomorphism locally in time.

(ii) It follows from (A.58) that

$$\hat{\mathbf{u}}(\chi(t, 0, \alpha), t) = \mathbf{u}_0(\alpha) + \int_0^t \nabla \phi(\chi(s, 0, \alpha), s) ds.$$

We take an inner product with $\chi(t, 0, \alpha), t$ to get

$$\begin{aligned} & \langle \hat{\mathbf{u}}(\chi(t, 0, \alpha), t), \chi(t, 0, \alpha) \rangle \\ &= \langle \mathbf{u}_0(\alpha), \chi(t, 0, \alpha) \rangle + \int_0^t \langle \nabla \phi(\chi(s, 0, \alpha), s), \chi(t, 0, \alpha) \rangle ds \\ (A.64) \quad &= \langle \mathbf{u}_0(\alpha), \chi(t, 0, \alpha) - \alpha \rangle + \langle \mathbf{u}_0(\alpha), \alpha \rangle + \int_0^t \langle \nabla \phi(\chi(s, 0, \alpha), s), \chi(t, 0, \alpha) \rangle ds. \end{aligned}$$

Since

$$\begin{aligned} |\chi(t, 0, \alpha) - \alpha| &= \left| \int_0^t \mathbf{v}(\chi(s, 0, \alpha), s) ds \right| \leq 6K_0 \delta^* T, \\ \text{and } |||\nabla \phi|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} &\leq R_3, \end{aligned}$$

Hence in (A.64), we have

$$\langle \hat{\mathbf{u}}(\chi(\alpha, t), t), \chi(\alpha, t) \rangle \leq 6K_0(\delta^*)^2 T - \eta_0 \|\alpha\|^2 + R_3 T.$$

Now we choose T sufficiently small so that

$$\left(6K_0(\delta^*)^2 + R_3\right)T \leq \frac{\eta_0 r_a^2}{2} \leq \frac{\eta_0}{2} \|\alpha\|^2, \quad \alpha \in \Omega_s(0).$$

Then we have

$$\langle \hat{\mathbf{u}}(\chi(\alpha, t), t), \chi(\alpha, t) \rangle \leq -\frac{\eta_0}{2} \|\alpha\|^2.$$

On the other hand, since

$$\frac{d}{ds} |\chi(0, 0, \alpha)|^2 \leq -2\eta_0 |\chi(0, 0, \alpha)|^2 = -2\eta_0 \|\alpha\|^2,$$

we have

$$|\chi(t, 0, \alpha)| \leq \|\alpha\| \quad \text{for } t \leq T \ll 1.$$

Therefore we obtain

$$\langle \hat{\mathbf{u}}(\chi(t, 0, \alpha), t), \chi(t, 0, \alpha) \rangle \leq -\frac{\eta_0}{2} |\chi(t, 0, \alpha)|^2.$$

□

A.1.7. A priori estimates for Step 7 of Section 7.2. Finally we prove the existence of a linear extension map and some estimates of the extension.

Lemma A.11. *Let $\mathcal{S}(t), t \in [0, T]$ be the $C^{2,\gamma}$ -regular simple closed convex curve in \mathbb{R}^2 provided by Lemma A.6 such that $\mathcal{S}(t)$ lies inside the annulus Ω_* and $\Omega_s(t; \mathbf{v})$ is the corresponding sheath region $0 \leq t \leq T$, T sufficiently small. Then there exists a bounded linear operator $\mathcal{E}(\cdot; \mathcal{S}(t)) : C^{2,\gamma}(\Omega_s(t; \mathbf{v})) \rightarrow C^{2,\gamma}(\Omega_1)$ satisfying*

- (a) $\mathcal{E}(\hat{\mathbf{u}}|\mathcal{S}(t)) = \hat{\mathbf{u}}$, in $\Omega_s(t; \mathbf{v})$,
- (b) $\mathcal{E}(\hat{\mathbf{u}}|\mathcal{S}(t))$ has support in $B(0, 3\delta^*)$,
- (c) $|||\mathcal{E}(\hat{\mathbf{u}}|\mathcal{S}(t))|||_{2,\gamma,\bar{\Omega}_1} \leq K_0 |||\hat{\mathbf{u}}|||_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})}$,

where K_0 is independent of $t \in [0, T]$ and Ω_* is the annulus region (??).

Proof. Since the proof is rather long, we delay its proof until Appendix C. □

A.2. Continuity of \mathcal{F} . In this part, we establish the continuity of \mathcal{F} which in turn imply the existence of a fixed point of \mathcal{F} .

Lemma A.12. *Let f be a continuous function such that*

$$\begin{aligned} f(t) &\leq C_0 + C_1(f(t))^2, \quad t \geq 0, \\ f(0) &\leq C_0 \quad \text{and} \quad C_0 C_1 \leq \frac{1}{8}, \end{aligned}$$

where C_0 and C_1 are positive constants independent of t . Then we have

$$f(t) \leq 2C_0.$$

Proof. Define

$$F(k) = C_1 k^2 - k + C_0.$$

Then by direct calculation, we have

$$\min F(k) = \frac{-1 + 2C_0 C_1}{2C_1} < 0 \quad \text{at } k = \frac{1}{2C_1}.$$

Now we denote r_1 and r_2 by the roots of $F(k) = 0$ such that $r_1 < r_2$. Then by direct calculation, the smallest root r_1 satisfies

$$C_0 \leq r_1 = \frac{2C_0}{1 + \sqrt{1 - 4C_0 C_1}} \leq 4(\sqrt{2} - 1)C_0 \leq 2C_0.$$

On the other hand, since $F(f(t)) \geq 0$, we have two cases:

$$\text{either } f \leq r_1 \quad \text{or} \quad f \geq r_2,$$

however since $f(0) \leq C_0 \leq r_1$ and $f(t)$ is continuous, we have

$$f \leq r_1 \leq 2C_0.$$

□

Proposition A.1. *There exists a positive constant T such that the map \mathcal{F} with $\mathcal{F}(\mathbf{v}) := \mathcal{E}(\hat{\mathbf{u}}; \mathcal{S}(t))$ is a well-defined map from $\mathcal{B}(T)$ to $\mathcal{B}(T)$.*

Proof. For the time being, we assume T sufficiently small so that

$$(A.65) \quad T \leq \min\{T_1, T_2\}.$$

So all estimates in the previous lemmas hold.

(i) By the construction of $\hat{\mathbf{u}}$ in the sheath region $\Lambda_s(T; \mathbf{v})$, we have from solving (A.57) along the characteristic

$$(A.66) \quad \hat{\mathbf{u}}(\mathbf{x}, t) = \begin{cases} \mathbf{u}_0(\boldsymbol{\alpha}(\mathbf{x}, t)) + \int_0^t \nabla \phi(\hat{\boldsymbol{\chi}}(s, t, \mathbf{x}), s) ds & t_0 = 0, \\ -\boldsymbol{\nu}(\boldsymbol{\alpha}(\mathbf{x}, t)) + \int_{t_0}^t \nabla \phi(\hat{\boldsymbol{\chi}}(s, t, \mathbf{x}), s) ds & t_0 > 0. \end{cases}$$

In (A.66), we have

$$\|\hat{u}_i\|_{0, \bar{\Lambda}_s(T; \mathbf{v})} \leq \begin{cases} \|u_{i0}\|_{0, \bar{\Omega}_s(0)} + t \|\nabla \phi\|_{0, \bar{\Lambda}_s(T; \mathbf{v})} & t_0 = 0, \\ 1 + (t - t_0) \|\nabla \phi\|_{0, \bar{\Lambda}_s(T; \mathbf{v})} & t_0 > 0. \end{cases}$$

This of course implies

$$(A.67) \quad \begin{aligned} \|\hat{u}_i\|_{0, \bar{\Lambda}_s(T; \mathbf{v})} &\leq \|u_{i0}\|_{0, \bar{\Omega}_s(0)} + T \|\nabla \phi\|_{0, \bar{\Lambda}_s(T; \mathbf{v})} \\ &\leq \|u_{0i}\|_{0, \bar{\Omega}_s(0)} + TR_3 \quad \text{by Lemma A.8.} \end{aligned}$$

On the other hand, we use Lemma A.2 to obtain

$$(A.68) \quad [[\hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq [u_{0i}]_{0,\gamma,\bar{\Omega}_s(0)} + C_1(T)R_3.$$

We combine (A.67) and (A.68) to get

$$(A.69) \quad \|\hat{u}_i\|_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq \|u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + TR_3 + C_1(T)R_3.$$

We assume T sufficiently small so that

$$(A.70) \quad TR_3 + C_1(T)R_3 \leq \frac{\delta^*}{3}.$$

Here we used $C_1(T) = \mathcal{O}(T^{1-\gamma})$. Hence we have from (A.70) that

$$(A.71) \quad \max_{i=1,2} \|\hat{u}_i\|_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq \max_{i=1,2} \|u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}.$$

(ii) We differentiate the momentum equation

$$\partial_t \hat{\mathbf{u}} + \hat{\mathbf{u}} \cdot \nabla \hat{\mathbf{u}} = \nabla \phi$$

with respect to x_k to find

$$(A.72) \quad \frac{D(\partial_{x_k} \hat{u}_i)}{Dt} + \sum_{j=1}^3 (\partial_{x_k} \hat{u}_j)(\partial_{x_i} \hat{u}_j) = \partial_{x_k}(\partial_{x_i} \phi),$$

where $\frac{D}{Dt} = \partial_t + \hat{\mathbf{u}} \cdot \nabla$. We integrate (A.72) along the characteristic to get

(i) if $t_0 = 0$,

$$\begin{aligned} \partial_{x_k} \hat{u}_i(\mathbf{x}, t) &= \partial_{x_k} u_{0i}(\boldsymbol{\alpha}(\mathbf{x}, t)) - \sum_{j=1}^2 \int_0^t \left((\partial_{x_k} \hat{u}_j)(\partial_{x_i} \hat{u}_j) \right) (\hat{\chi}(s, t, \mathbf{x}), s) ds \\ &\quad + \int_0^t \partial_{x_k}(\partial_{x_i} \phi)(\hat{\chi}(s, t, \mathbf{x}), s) ds; \end{aligned}$$

(ii) and if $t_0 > 0$,

$$\begin{aligned} \partial_{x_k} \hat{u}_i(\mathbf{x}, t) &= -\partial_{x_k} \nu_i(\boldsymbol{\alpha}(\mathbf{x}, t)) - \sum_{j=1}^2 \int_{t_0}^t \left((\partial_{x_k} \hat{u}_j)(\partial_{x_i} \hat{u}_j) \right) (\hat{\chi}(s, t, \mathbf{x}), s) ds \\ &\quad + \int_{t_0}^t \partial_{x_k}(\partial_{x_i} \phi)(\hat{\chi}(s, t, \mathbf{x}), s) ds. \end{aligned} \quad (A.73)$$

The above equalities yield

$$\begin{aligned} \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha \hat{u}_i\|_{0,\bar{\Lambda}_s(T;\mathbf{v})} &\leq \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\bar{\Omega}_s(0)} + 2T \left(\max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha \hat{u}_i\|_{0,\bar{\Lambda}_s(T;\mathbf{v})} \right)^2 \\ &\quad + T \max_{|\alpha|=2} \|\partial^\alpha \phi\|_{0,\bar{\Lambda}_s(T;\mathbf{v})}, \quad 0 \leq t \leq T. \end{aligned}$$

Since $T \ll 1$, it follows from Lemma A.12 that

$$\begin{aligned} \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha \hat{u}_i\|_{0,\bar{\Lambda}_s(T;\mathbf{v})} &\leq 2 \left(\max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\bar{\Omega}_s(0)} + T \max_{|\alpha|=2} \|\partial^\alpha \phi\|_{0,\bar{\Lambda}_s(T;\mathbf{v})} \right) \\ (A.74) \quad &\leq 2 \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\bar{\Omega}_s(0)} + 2TR_3. \end{aligned}$$

On the other hand, it follows from the inequalities (A.73) and (A.2) that

$$\begin{aligned} & \max_{i=1,2} \max_{|\alpha|=1} [[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \\ & \leq \max_{i=1,2} \max_{|\alpha|=1} [\partial^\alpha u_{0i}]_{0,\gamma,\bar{\Omega}_s(0)} + 4C_1(T) \left(2 \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha \hat{u}_{0i}\|_{0,\bar{\Omega}_s(0)} + 2TR_3 \right) \\ & \times \max_{i=1,2} \max_{|\alpha|=1} [[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} + C_1(T)R_3. \end{aligned}$$

We assume that

$$4C_1(T) \left(2 \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\bar{\Lambda}_s(T;\mathbf{v})} + 2TR_3 \right) \leq \frac{1}{2}.$$

Here we used $C_1(T) = \mathcal{O}(T^{1-\gamma})$.

Then we have

$$(A.75) \quad \max_{i=1,2} \max_{|\alpha|=1} [[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq 2 \max_{i=1,2} \max_{|\alpha|=1} [[\nabla u_{0i}]]_{0,\gamma,\bar{\Omega}_s(0)} + 2C_1(T)R_3.$$

We combine (A.74) and (A.75) to get

$$\max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha \hat{u}_i\|_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq 2 \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + 2TR_3 + 2C_1(T)R_3.$$

We assume again that T is sufficiently small so that

$$(A.76) \quad 2TR_3 + 2C_1(T)R_3 \leq \frac{\delta^*}{3}.$$

Then we have

$$(A.77) \quad \max_{i=1,2} \|\nabla \hat{u}_i\|_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq 2 \max_{i=1,2} \|\nabla u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}.$$

(iii) We differentiate (A.72) with respect to x_l to obtain

$$\begin{aligned} & \frac{D(\partial_{x_k x_l}^2 \hat{u}_i)}{Dt} + \sum_{j=1}^3 \left[(\partial_{x_l} \hat{u}_j)(\partial_{x_j x_k}^2 \hat{u}_i) + (\partial_{x_i} \hat{u}_j)(\partial_{x_k x_l}^2 \hat{u}_j) + (\partial_{x_k} \hat{u}_j)(\partial_{x_i x_l}^2 \hat{u}_j) \right] \\ (A.78) \quad & = \partial_{x_k x_l}^2 (\partial_{x_i} \phi). \end{aligned}$$

We integrate the equation (A.78) along the characteristic curve to find

$$\begin{aligned}
& (i) \text{ if } t_0 = 0, \\
& \partial_{x_k x_l}^2 \hat{u}_i(\mathbf{x}, t) = \partial_{x_k x_l}^2 u_{0i}(\boldsymbol{\alpha}(\mathbf{x}, t)) \\
& \quad - \sum_{j=1}^2 \int_0^t \left((\partial_{x_l} \hat{u}_j)(\partial_{x_j x_k}^2 \hat{u}_i) + (\partial_{x_i} \hat{u}_j)(\partial_{x_k x_l}^2 \hat{u}_j) + (\partial_{x_k} \hat{u}_j)(\partial_{x_i x_l}^2 \hat{u}_j) \right) (\hat{\chi}(s, t, \mathbf{x}), s) ds \\
& \quad + \int_0^t \partial_{x_k} \partial_{x_l} (\partial_{x_i} \phi)(\hat{\chi}(s, t, \mathbf{x}), s) ds; \\
& (ii) \text{ if } t_0 > 0, \\
& \partial_{x_k} \partial_{x_l} \hat{u}_i(\mathbf{x}, t) = -\partial_{x_k x_l}^2 \nu_i(\boldsymbol{\alpha}(\mathbf{x}, t)) \\
& \quad - \sum_{j=1}^2 \int_{t_0}^t \left((\partial_{x_l} \hat{u}_j)(\partial_{x_j x_k}^2 \hat{u}_i) + (\partial_{x_i} \hat{u}_j)(\partial_{x_k x_l}^2 \hat{u}_j) + (\partial_{x_k} \hat{u}_j)(\partial_{x_i x_l}^2 \hat{u}_j) \right) (\hat{\chi}(s, t, \mathbf{x}), s) ds \\
& \quad + \int_{t_0}^t \partial_{x_k x_l}^2 (\partial_{x_i} \phi)(\hat{\chi}(s, t, \mathbf{x}), s) ds.
\end{aligned}
\tag{A.79}$$

Then it follows from (A.79) that

$$\begin{aligned}
& \max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha \hat{u}_i|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} \\
& \leq \max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha \hat{u}_{0i}|||_{0, \bar{\Omega}_s(0)} + 6T \left(2 \max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha u_{0i}|||_{0, \gamma, \bar{\Omega}_s(0)} + \frac{\delta^*}{3} \right) \\
& \quad \times \max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha \hat{u}_i|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} + TR_3.
\end{aligned}$$

We choose T sufficiently small so that

$$6T \left(2 \max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha u_{0i}|||_{0, \gamma, \bar{\Omega}_s(0)} + \frac{\delta^*}{3} \right) \leq \frac{1}{2} \quad \text{and} \quad TR_3 \leq \frac{\delta^*}{12}.
\tag{A.80}$$

Then we have

$$\max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha \hat{u}_i|||_{0, \bar{\Lambda}_s(T; \mathbf{v})} \leq 2 \max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha u_{0i}|||_{0, \bar{\Omega}_s(0)} + \frac{\delta^*}{6}.
\tag{A.81}$$

We need to check the Hölder seminorm of $\partial^\alpha \hat{u}_i$. Again we use (A.79) to find

$$\begin{aligned}
& \max_{i=1,2} \max_{|\alpha|=2} [[\partial^\alpha \hat{u}_i]]_{0, \gamma, \bar{\Lambda}_s(T; \mathbf{v})} \\
& \leq \max_{i=1,2} [[\partial^\alpha u_{0i}]]_{0, \gamma, \bar{\Lambda}_s(T; \mathbf{v})} + 6C_1(T) \left(2 \max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha u_{0i}|||_{0, \gamma, \bar{\Omega}_s(0)} + \frac{\delta^*}{3} \right) \\
& \quad \times \max_{i=1,2} \max_{|\alpha|=2} [[\partial^\alpha \hat{u}_i]]_{0, \gamma, \bar{\Lambda}_s(T; \mathbf{v})} + 6C_1(T) \left(2 \max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha u_{0i}|||_{0, \gamma, \bar{\Omega}_s(0)} + \frac{\delta^*}{3} \right) \\
& \quad \times \left(2 \max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha u_{0i}|||_{0, \bar{\Omega}_s(0)} + \frac{\delta^*}{6} \right) + C_1(T)R_3.
\end{aligned}$$

Here we have used (A.2).

We assume that T sufficiently is sufficiently small that

$$(A.82) \quad 6C_1(T) \left(2 \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3} \right) \leq \frac{1}{2} \quad \text{and}$$

$$(A.83) \quad 6C_1(T) \left(2 \max_{i=1,2} \max_{|\alpha|=1} \|\partial^\alpha u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3} \right) \\ \times \left(2 \max_{i=1,2} \max_{|\alpha|=2} \|\partial^\alpha \hat{u}_{0i}\|_{0,\Omega_s(0)} + \frac{\delta^*}{6} \right) + C_1(T)R_3 \leq \frac{\delta^*}{12}.$$

Hence we have

$$(A.84) \quad \max_{i=1,2} \max_{|\alpha|=2} [[\partial^\alpha \hat{u}_i]]_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq 2 \max_{i=1,2} \max_{|\alpha|=2} [\partial^\alpha u_{0i}]_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{6}.$$

We combine (A.81) and (A.84) to get

$$(A.85) \quad \max_{i=1,2} \max_{|\alpha|=2} |||\partial^\alpha \hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq 2 \max_{i=1,2} \max_{|\alpha|=2} \|\partial^\alpha u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + \frac{\delta^*}{3}.$$

We combine (A.71), (A.77) and (A.85) to get

$$(A.86) \quad \max_{i=1,2} \sum_{0 \leq k \leq 2} \max_{|\alpha|=k} |||\partial^\alpha \hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \leq 2 \max_{i=1,2} \sum_{0 \leq k \leq 2} \max_{|\alpha|=k} \|\partial^\alpha u_{0i}\|_{0,\gamma,\bar{\Omega}_s(0)} + \delta^* \leq 3\delta^*.$$

(iv) It follows from the Burgers' equation (A.57) that

$$(A.87) \quad \begin{aligned} \max_{i=1,2} |||\partial_t \hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} &\leq 2 \left(\max_{i=1,2} |||\hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \right) \left(\max_{i=1,2} \max_{|\alpha|=1} |||\partial^\alpha \hat{u}_i|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \right) \\ &\quad + |||\nabla \phi|||_{0,\gamma,\bar{\Lambda}_s(T;\mathbf{v})} \\ &\leq 18(\delta^*)^2 + R_3. \end{aligned}$$

Finally we combine all estimates (A.86) and (A.87) to get

$$\max_{i=1,2} \left(\max_{|\alpha| \leq 2} |||\partial^\alpha \hat{u}_i|||_{0,\gamma,\bar{\Lambda}(T)} \right) \leq 3\delta^* \quad \text{and} \quad \max_{i=1,2} |||\partial_t \hat{u}_i|||_{0,\gamma,\bar{\Lambda}(T)} \leq \left(18(\delta^*)^2 + R_3 \right).$$

By the construction of extension of $\hat{\mathbf{u}}$, we find

$$(A.88) \quad (a) \quad \max_{i=1,2} \left(\max_{|\alpha| \leq 2} |||\partial^\alpha u_i|||_{0,\gamma,\bar{\Lambda}(T)} \right) \leq 3K_0\delta^*,$$

$$(A.89) \quad (b) \quad \max_{i=1,2} |||\partial_t u_i|||_{0,\gamma,\bar{\Lambda}(T)} \leq K_0 \left(18(\delta^*)^2 + R_3 \right).$$

Here we notice that the norm $\|\cdot\|_{0,\gamma,\bar{\Omega}_s(t;\mathbf{v})}$ in Appendix C can be generalized to the space-time norm $|||\cdot|||_{0,\gamma,\bar{\Lambda}(T)}$.

Finally the estimates (A.88) and (A.89) show that $\mathbf{u} \in \mathcal{B}(T)$. □

We set

$\Lambda_s(T; r)$: the sheath region determined by the interface r ,

and recall that an interface $\mathcal{S}(t)$ is represented by the radial function $r(\cdot, t)$.

Lemma A.13. ([13]) *Let $\tau < \gamma$,*

$$\begin{aligned} r_i &\rightarrow r \quad \text{in } C^{1,\tau}(\mathbb{R} \times [0, T]) \quad \text{as given by Lemma A.5} \quad \text{and} \\ \hat{\mathbf{u}}_i &\in C^{1,\tau}(\Lambda_s(T; r_i)) : \text{ be associated solutions of (A.57) for each } i \text{ as given by Lemma A.10.} \end{aligned}$$

Let $\mathcal{E}(\hat{\mathbf{u}}_i(\cdot, t); r_i(t))$ be the extension of $\hat{\mathbf{u}}_i(\cdot, t)$ with

$$\mathcal{E}(\mathbf{u}_i(\cdot, t); r_i(t)) \rightarrow \mathbf{w} \quad \text{in } C^{1,\tau}(\bar{\Lambda}(T)).$$

Then we have

$$\mathbf{w} = \mathcal{E}(\hat{\mathbf{w}} \Big|_{\Lambda_s(T; r)}).$$

Proof. The proof follows from a straightforward modification of the proof in [13] as hence is omitted. \square

Proof of Theorem 7.3

Let $\{\mathbf{v}_i\}$ be a convergent sequence in \mathcal{B}_T in the topology of \mathcal{T} (see (??)) such that

$$\mathbf{v}_i \rightarrow \mathbf{v} \quad \text{in } C^{1,\tau}(\bar{\Lambda}(T)) \quad \text{and} \quad \partial^\alpha \mathbf{v}_i \rightarrow \partial^\alpha \mathbf{v}, \quad \text{in } C^{0,\tau} \quad |\alpha| = 2, \quad 0 < \tau < \gamma.$$

By Proposition A.1, $\mathcal{F}(\mathbf{v}_i)$ is well-defined as an element of $\mathcal{B}(T)$ for each i and the sequence $\{\mathcal{F}(\mathbf{v}_i)\}$ is uniformly bounded in \mathcal{T} . Since the Arzela-Ascoli theorem implies the compact imbedding of $C^{1,\gamma}(\bar{\Lambda}(T))$ into $C^{1,\tau}(\bar{\Lambda}(T))$ with $0 < \tau < \gamma$, we have a convergent subsequence which we still denote by $(\mathbf{v}_i, \mathcal{F}(\mathbf{v}_i))$:

$$\mathcal{F}(\mathbf{v}_i) \rightarrow \mathbf{w} \quad \text{in } C^{1,\tau}(\bar{\Lambda}(T)).$$

We claim:

$$(A.90) \quad \mathcal{F}(\mathbf{v}) = \mathbf{w}.$$

Proof of the claim: Let $(\chi_i, n_i, \hat{\mathbf{u}}_i, \mathcal{S}_i, \phi_i)$ and $(\chi, n, \hat{\mathbf{u}}, \mathcal{S}, \phi)$ be the quantities corresponding to \mathbf{v}_i and \mathbf{v} respectively.

Step I. Suppose that

$$\mathbf{v}_i \rightarrow \mathbf{v} \quad \text{in } C^{1,\tau}(\bar{\Lambda}(T)) \quad \text{as } i \rightarrow \infty.$$

Then it follows from Lemma A.1 (2) that $\chi_i(\cdot, t, \mathbf{x}) \rightarrow \chi(\cdot, t, \mathbf{x}) \quad \text{in } C^{1,\gamma}([0, T])$ as $i \rightarrow \infty$ and hence since $\mathbf{v} \in C^{1,\tau}(\bar{\Lambda}(T))$, we have

$$\nabla \cdot \mathbf{v}_i(\chi_i(\xi, t, \mathbf{x}), \xi) \rightarrow \nabla \cdot \mathbf{v}(\chi(\xi, t, \mathbf{x}), \xi) \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0, T]) \quad \text{as } i \rightarrow \infty.$$

We use Lemma A.2 to get

$$(A.91) \quad \int_0^t \nabla \cdot \mathbf{v}_i(\chi_i(\xi, t, \mathbf{x}), \xi) d\xi \rightarrow \int_0^t \nabla \cdot \mathbf{v}(\chi(\xi, t, \mathbf{x}), \xi) d\xi \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0, T]) \quad \text{as } i \rightarrow \infty.$$

On the other hand, since $\alpha_i \rightarrow \alpha$ in $C^{1,\tau}(\partial\Omega_0 \times [0, T])$ as $i \rightarrow \infty$, we have

$$(A.92) \quad n_0(\alpha_i(\mathbf{x}, t)) \rightarrow n_0(\alpha(\mathbf{x}, t)) \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0, T]) \quad \text{as } i \rightarrow \infty.$$

Here we used the fact that n_0 is in $C^{1,\tau}(\mathbb{R}^2)$. Recall the formula for n :

$$n(\mathbf{x}, t) = n_0(\alpha(\mathbf{x}, t)) \exp \left(- \int_0^t (\nabla \cdot \mathbf{v})(\chi(\xi, 0, \alpha(\mathbf{x}, t)), \xi) d\xi \right).$$

We now combine (A.91) and (A.92) and the above formula to see

$$n_i(\mathbf{x}, t) \rightarrow n(\mathbf{x}, t) \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0, T]) \quad \text{as } i \rightarrow \infty,$$

which in turn implies

$$h_{0i} = \partial_t g - (n_i \mathbf{v}_i) \cdot \boldsymbol{\nu} \rightarrow h_0 = \partial_t g - (n \mathbf{v}) \cdot \boldsymbol{\nu}, \quad \text{in } C^{1,\tau}(\partial\Omega_0 \times [0, T]).$$

By Lemma A.5 and Lemma A.6, we have

$$(A.93) \quad \nabla \zeta_i \rightarrow \nabla \zeta \quad \text{in } C^{1,\tau}(\bar{\Omega}_* \times [0, T]) \quad \text{as } i \rightarrow \infty,$$

$$(A.94) \quad (\theta_i, r_i, n_i) \rightarrow (\theta, r, n) \quad \text{in } C^{1,\tau}(\mathbb{R} \times [0, T]).$$

Step II. Let $\Lambda_s(T; \mathbf{v})$ be the sheath region determined by \mathbf{v} . Since $\mathcal{F}(\mathbf{v})$ is uniquely determined by \mathbf{v} on the sheath region, once we can show \mathbf{w} satisfies equations (??)-(??) and the interface conditions:

$$(A.95) \quad \mathbf{u} = -\boldsymbol{\nu} \quad \text{and} \quad \nabla \phi \cdot \boldsymbol{\nu} = 0 \quad \text{on } \mathcal{S}(t),$$

for the orthogonal flow in the sheath region $\Lambda_s(T; \mathbf{v})$, we will have

$$\mathbf{w} = \mathcal{F}(\mathbf{v}) \quad \text{in } \Lambda_s(T; \mathbf{v}).$$

So let us proceed to show that \mathbf{w} satisfies the sheath system (??) and boundary conditions (A.95) in $\Lambda_s(T; \mathbf{v})$. Let \mathcal{O} be any open set compactly supported in $\Lambda_s(T; \mathbf{v})$. Then by (A.93) and (A.94), since $r_i \rightarrow r$ in $C^{1,\tau}(\mathbb{R} \times [0, T])$, we have

$$\mathcal{O} \subset \Lambda_s(T; \mathbf{v}_i) \quad i \geq N.$$

For $i \geq N$, we know that $(n_i, \mathbf{v}_i, \phi_i, \hat{\mathbf{u}}_i)$ satisfy

$$\begin{cases} \partial_t n_i + \nabla \cdot (n_i \mathbf{v}_i) = 0, & (\mathbf{x}, t) \in \mathcal{O}, \\ \Delta \phi_i = n_i, \\ \partial_t \hat{\mathbf{u}}_i + (\hat{\mathbf{u}}_i \cdot \nabla) \hat{\mathbf{u}}_i = \nabla \phi_i, \end{cases}$$

and

$$(n_i, \mathbf{v}_i, \phi_i, \hat{\mathbf{u}}_i) \rightarrow (n, \mathbf{v}, \phi, \mathbf{w}) \quad \text{in } C^{1,\tau}(\bar{\mathcal{O}}),$$

and hence we find in the limit as $i \rightarrow \infty$,

$$\begin{cases} \partial_t n + \nabla \cdot (n \mathbf{v}) = 0, & (\mathbf{x}, t) \in \mathcal{O}, \\ \Delta \phi = n, \\ \partial_t \mathbf{w} + (\mathbf{w} \cdot \nabla) \mathbf{w} = \nabla \phi. \end{cases}$$

Next we check the boundary conditions on the sheath interface. Since by (A.94) $\theta_i \rightarrow \theta$ in $C^{1,\gamma}(\mathbb{R})$ and $\boldsymbol{\nu}_i = (\cos \theta_i, \sin \theta_i)$, we obtain

$$\boldsymbol{\nu}_i \rightarrow \boldsymbol{\nu}, \quad \text{in } C^{1,\gamma}(\mathbb{R} \times [0, T]).$$

On the other hand, we have

$$\nabla \phi_i \cdot \boldsymbol{\nu}_i = 0 \quad \text{and} \quad \hat{\mathbf{u}}_i = -\boldsymbol{\nu}_i \quad \text{on } \mathcal{S}_i.$$

Letting $i \rightarrow \infty$, we see

$$\nabla \phi \cdot \boldsymbol{\nu} = 0 \quad \text{and} \quad \mathbf{w} = -\boldsymbol{\nu} \quad \text{on } \mathcal{S}.$$

Hence we have shown that \mathbf{w} satisfies the sheath system (??) in the sheath region and boundary conditions (A.95). By the uniqueness of the construction, we have

$$\mathcal{F}(\mathbf{v}) = \mathbf{w} \quad \text{on } \Lambda_s(T; \mathbf{v}).$$

Step III. Recall by (A.94) that we have

$$r_i \rightarrow r \quad \text{in } C^{1,\tau}(\mathbb{R} \times [0, T]) \quad \text{and} \quad \mathcal{F}(\mathbf{v}_i) \rightarrow \mathbf{w}.$$

Then by Lemma A.12, we have

$$\mathbf{w} = \mathcal{E}(\mathbf{w} \Big|_{\Lambda_s(T; \mathbf{v})}) = \mathcal{F}(\mathbf{v}).$$

Hence we showed that \mathcal{F} is continuous in the $C^{1,\tau}$ -topology. Since \mathcal{F} is a continuous map on the compact and convex set $\mathcal{B}(T)$ of $C^{1,\gamma}$ space, by the Schauder fixed point theorem, \mathcal{F} has a fixed point \mathbf{u} such that

$$\mathcal{F}(\mathbf{u}) = \mathbf{u}.$$

This \mathbf{u} is a desired smooth solution of the sheath system. This completes the proof.

APPENDIX B. **Gronwall-Bellman type inequality**

In this appendix, we prove the Gronwall-Bellman type inequality.

Let f be a real valued positive continuous function and suppose a nonnegative real valued function y satisfies the following integral inequality:

$$y(t) \leq f(t) + c^2 \int_0^t \int_0^{t_1} y(\tau) d\tau dt_1.$$

Then y satisfies

$$\begin{aligned} y(t) &\leq f(t) + c^2 \int_0^t \int_0^{t_1} f(s) \exp[c(t - 2t_1 + s)] ds dt_1 \\ &= f(t) + \left(\max_{\tau \in [0, t]} f(\tau) \right) \mathcal{O}(t^2), \quad \text{as } t \rightarrow 0. \end{aligned}$$

Proof. Let us set

$$w(t) \equiv \int_0^t \int_0^{t_1} y(\tau) d\tau dt_1.$$

Then we have

$$(B.96) \quad y(t) \leq f(t) + c^2 w(t).$$

It is easy to see that

$$w''(t) = y(t), \quad w(0) = 0, \quad w'(0) = 0.$$

In (C.97), we have a differential inequality for w :

$$w''(t) \leq c^2 w(t) + f(t).$$

Now we introduce another dependent variable u defined by

$$w(t) = \exp(ct)u(t).$$

By direct calculation, we obtain a differential inequality for u :

$$u'' + 2cu' \leq f(t) \exp(-ct).$$

We multiply an integrating factor $\exp(2ct)$ to get

$$\left(\exp(2ct)u' \right)' \leq f(t) \exp(ct).$$

Next we integrate the above inequality to get

$$u(t) \leq \int_0^t \int_0^{t_1} f(s) \exp[-c(2t_1 - s)] ds dt_1,$$

where we used

$$u(0) = 0, \quad u'(0) = 0.$$

This implies

$$\begin{aligned} w(t) &= \exp(ct)u(t) \\ &\leq \int_0^t \int_0^{t_1} f(s) \exp[c(t - 2t_1 + s)] ds dt_1. \end{aligned}$$

In (B1), we have

$$\begin{aligned}
y(t) &\leq f(t) + c^2 \int_0^t \int_0^{t_1} f(s) \exp[c(t - 2t_1 + s)] ds dt_1 \\
&\leq f(t) + c^2 \left(\max_{\tau \in [0, t]} f(\tau) \right) \int_0^t \int_0^{t_1} \exp[c(t - 2t_1 + s)] ds dt_1 \\
&= f(t) + \left(\max_{\tau \in [0, t]} f(\tau) \right) \mathcal{O}(t^2) \quad \text{as } t \rightarrow 0,
\end{aligned}$$

where we used

$$\int_0^t \int_0^{t_1} \exp[c(t - 2t_1 + s)] ds dt_1 = \frac{1}{c^2} \left[-1 + \frac{1}{2}(e^{-ct} + e^{ct}) \right] = \frac{\mathcal{O}(t^2)}{c^2} \quad \text{as } t \rightarrow 0.$$

□

APPENDIX C. **Extension Theorem**

In this part, we present an extension theorem for $C^{2,\gamma}$ -functions defined on the sheath region $\Omega_s(t; \mathbf{v})$, $t \in [0, T]$ to the bigger domain $\Omega_1 := B(0, 3\delta^*) - \Omega_0$.

We first consider an upper bounds for the length of a convex polygon and a simple closed convex curve inside the annulus $A(r_1, r_2)$ defined by

$$A(r_1, r_2) := \{\mathbf{x} \in \mathbb{R}^2 : r_1 < |\mathbf{x}| < r_2\}.$$

Lemma C.1. *Let \mathcal{P} and \mathcal{C} be a convex n -polygon and a convex curve inside the annulus $A(r_1, r_2)$ respectively. Then we have*

$$l(\mathcal{P}) \leq 2\pi r_2 \quad \text{and} \quad l(\mathcal{C}) \leq 2\pi r_2,$$

where $l(\mathcal{P})$ and $l(\mathcal{C})$ denote the lengths of the polygon \mathcal{P} and the curve \mathcal{C} respectively.

Proof. (i) Let $\mathcal{P} = \mathcal{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)$ be a convex n -polygon whose vertices are $\mathbf{x}_1, \dots, \mathbf{x}_n$. We choose any point \mathbf{c}_0 inside \mathcal{P} , and we set

\mathbf{y}_i : the intersection point with a ray $\overrightarrow{\mathbf{c}_0 \mathbf{x}_i}$ and a circle $B(0, r_2)$.

Then it is easy to see that

$$(C.97) \quad l(\mathcal{P}(\mathbf{x}_1, \dots, \mathbf{x}_n)) \leq l(\mathcal{P}(\mathbf{y}_1, \dots, \mathbf{y}_n)).$$

On the other hand we know that

$$(C.98) \quad l(\mathcal{P}(\mathbf{y}_1, \dots, \mathbf{y}_n)) \leq l(B(0, r_2)) = 2\pi r_2.$$

We combine (C.97) and (C.98) to obtain

$$l(\mathcal{P}) \leq 2\pi r_2.$$

(ii) Let \mathcal{C} be a simple closed convex curve lying inside $A(r_1, r_2)$. Note that for any simple closed convex curve there exists some polygon whose sides are parts of supporting lines of the given convex curve. Choose a sufficiently small positive constant $r_0 > 0$. Since the curve \mathcal{C} is compact, there exists a finite open cover of \mathcal{C} consisting of balls with a center $\bar{\mathbf{x}}_i$ and a radius r_0 , say,

$$\mathcal{C} \subset \cup_{i=1}^M B(\bar{\mathbf{x}}_i, r_0), \quad \text{where } \bar{\mathbf{x}}_i \in \mathcal{C}.$$

Consider an M -polygon consisting of parts of supporting lines at $\bar{\mathbf{x}}_i, i = 1, \dots, M$ and denote it by $\bar{\mathcal{P}}$. Then it follows from the result of (i) that

$$l(\mathcal{C}) \leq l(\bar{\mathcal{P}}) \leq 2\pi r_2.$$

□

Next we present the existence of a continuous linear extension operator from $C^{1,\gamma}(\Omega_s(t; \mathbf{v}))$ to $C^{1,\gamma}(\Omega_1)$. Even though the construction of this extension operator can be found in the literature, see for example [1, 28, 34], we slightly modify the proofs given in books [1, 28, 34] for our purpose.

The proof of Lemma A.10: We first consider the local extension near one generic point on the interface and then glue these local extensions together using the standard partition of unity to get a global extension. Let t be given.

Step I (local extension): Let \mathbf{x}_0 be any generic point on the interface $\mathcal{S}(t)$. Then there are two cases: either $\mathcal{S}(t)$ is flat near \mathbf{x}_0 , lying in the plane or it is not flat near \mathbf{x}_0 .

Case 1: $\mathcal{S}(t)$ is flat near \mathbf{x}_0 lying on some line.

For simplicity, we assume $\mathbf{x}_0 = (a_1, a_2)$ and the plane is $\{x_2 = a_2\}$. We choose an open ball $B(\mathbf{x}_0, r)$ such that

$$\begin{cases} B^+ := B(\mathbf{x}_0, r) \cap \{x_2 \geq a_2\} \subset B(0, 3\delta^*) - \Omega_s(t; \mathbf{v}), \\ B^- := B(\mathbf{x}_0, r) \cap \{x_2 \leq a_2\} \subset \bar{\Omega}_s(t; \mathbf{v}). \end{cases}$$

Let f be any $C^{2,\gamma}$ -function defined on $\Omega_s(t; \mathbf{v})$. We extend f to the ball $B^+ \cup B^-$ as follows.

$$\bar{f}(x_1, x_2) := \begin{cases} 6f(x_1, 2a_2 - x_2) - 8f(x_1, 3a_2 - 2x_2) + 3f(x_1, 4a_2 - 3x_2), & \text{if } (x_1, x_2) \in B^+, \\ f(x_1, x_2), & \text{if } (x_1, x_2) \in B^-. \end{cases}$$

Notice this choice of \bar{f} is not the same as given by Evans [28], since he only desired C^1 -regularity. We have used a special case of the result given in [1].

We claim: \bar{f} is $C^{2,\gamma}$ in the ball B .

We need to show all partial derivatives are continuous at $\mathbf{x}_0 = (a_1, a_2)$. Let us write $f^- := \bar{f}|_{B^-}$, $f^+ := \bar{f}|_{B^+}$. By direct calculation we obtain

$$\begin{aligned} & \bullet \partial_{x_1}^k f^-(x_1, x_2) \\ &= 6\partial_{x_1}^k f(x_1, 2a_2 - x_2) - 8\partial_{x_1}^k f(x_1, 3a_2 - 2x_2) + 3\partial_{x_1}^k f(x_1, 4a_2 - 3x_2), \quad k = 0, 1, 2, \\ & \bullet \partial_{x_2} f^-(x_1, x_2) \\ &= -6\partial_{x_2} f(x_1, 2a_2 - x_2) + 16\partial_{x_2} f(x_1, 3a_2 - 2x_2) - 9\partial_{x_2} f(x_1, 4a_2 - 3x_2), \\ & \bullet \partial_{x_2}^2 f^-(x_1, x_2) \\ &= 6\partial_{x_2}^2 f(x_1, 2a_2 - x_2) - 32\partial_{x_2}^2 f(x_1, 3a_2 - 2x_2) + 27\partial_{x_2}^2 f(x_1, 4a_2 - 3x_2), \\ & \bullet \partial_{x_1} \partial_{x_2} f^-(x_1, x_2) \\ &= -6\partial_{x_1} \partial_{x_2} f(x_1, 2a_2 - x_2) + 16\partial_{x_1} \partial_{x_2} f(x_1, 3a_2 - 2x_2) - 9\partial_{x_1} \partial_{x_2} f(x_1, 4a_2 - 3x_2) \end{aligned}$$

Now evaluate the above identities on the line $\{x_2 = a_2\}$ to see that extended function \bar{f} is C^2 in the ball B and we have

$$\begin{aligned} [\partial_{x_1}^2 f^-]_{0,\gamma,\bar{B}^+} &\leq 31[\partial_{x_1}^2 f]_{0,\gamma,\bar{B}^-}, & [\partial_{x_1} \partial_{x_2} f^-]_{0,\gamma,\bar{B}^+} &\leq 119[\partial_{x_1} \partial_{x_2} f]_{0,\gamma,\bar{B}^-} \\ \text{and } [\partial_{x_2}^2 f^-]_{0,\gamma,\bar{B}^+} &\leq 151[\partial_{x_2}^2 f]_{0,\gamma,\bar{B}^-}. \end{aligned}$$

Hence we have

$$\|\bar{f}\|_{2,\gamma,\bar{B}} \leq 151\|f\|_{1,\gamma,\bar{B}^-}.$$

Case 2: $\mathcal{S}(t)$ is not flat near \mathbf{x}_0 .

Since the interface $\mathcal{S}(t)$ is $C^{2,\gamma}$ -regular, we can find a $C^{2,\gamma}$ -mapping Φ with inverse Φ^{-1} such that Φ straightens out $\mathcal{S}(t)$ near \mathbf{x}_0 . We write $\mathbf{y} = \Phi(\mathbf{x})$, $f'(\mathbf{y}) := f(\Phi^{-1}(\mathbf{y}))$. We choose a small ball B as before. Then as in *Case 1*, we extend f' from B^- to B and get

$$\|\bar{f}'\|_{2,\gamma,\bar{B}} \leq 151\|f'\|_{1,\gamma,\bar{B}^-}.$$

Let $W := \Phi^{-1}(B)$ and $W^\pm := \Phi^{-1}(B^\pm)$. Then we have

$$\|\bar{f}\|_{2,\gamma,\bar{W}} \leq 151\|f\|_{2,\gamma,\bar{W}^-}.$$

Now we glue local extensions together using the partition of unity to get a global extension.

Step II (Global extension): We will extend f defined on $\Omega_s(t; \mathbf{v})$ to the bigger domain Ω_1 such that the extended \bar{f} has support in Ω_1 . Let r_1 be a sufficiently small number satisfying

$$0 < r_1 < \min \left\{ \delta^*, 0.5\delta_{*2} - r_b \right\}.$$

Then for such r_1 , we choose points $\mathbf{x}_i (i = 1, \dots, M(t))$ on the curve $\mathcal{S}(t)$ such that neighboring \mathbf{x}_i 's are located by the part of curve with length r except one pair of points, i.e.,

$$\begin{aligned} l(\text{part of an interface curve connecting } \mathbf{x}_i \text{ and } \mathbf{x}_{i+1}) &= r_0, \quad i = 1, \dots, M(t) - 1, \\ l(\text{part of an interface curve connecting } \mathbf{x}_{M(t)} \text{ and } \mathbf{x}_1) &\leq r. \end{aligned}$$

Then the number $M(t)$ of such points are bounded by

$$M(t) \leq \left\lceil \frac{l(\mathcal{S}(t))}{r_1} \right\rceil + 1,$$

where the bracket is the greatest integer function. Then by Lemma C.1, we know that

$$M(t) \leq \left\lceil \frac{4\pi\delta^*}{r_1} \right\rceil + 1, \quad t \in [0, T_*].$$

As in *Step I*, we extend f to $B(\mathbf{x}_i, r)$ for each i , and denote \bar{f}_i by the extended function. Now take an open set W_0 whose closure is a compact subset of $\Omega_s(t)$ and $\Omega_s(t) \subset W_0(t) \cup \left(\bigcup_{i=0}^{M(t)} W_i(t) \right)$. Let $\{\kappa_i\}$ be a partition of unity corresponding to the open covering $\{W_i(t)\}_{i=0}^{M(t)}$ of $\Omega_s(t; \mathbf{v})$ and define

$$\bar{f} := \sum_{i=0}^{M(t)} \kappa_i \bar{f}_i, \quad \bar{f}_0 = f.$$

It follows from *Step I* that

$$\|\bar{f}\|_{2,\gamma,\bar{\Omega}_1} \leq 151(M(t) + 1)\|f\|_{2,\gamma,\bar{\Omega}_s(t;\mathbf{v})}.$$

We take K_0 to be $151\left(\left\lceil \frac{4\pi\delta^*}{r_1} \right\rceil + 2\right)$ to obtain the desired result.

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